UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: PROBABLITY AND RANDOM PROCESSES

Problem Set 8
Fall 2006

Issued: Thursday, October 26, 2006 **Due:** Friday, November 3, 2006

Reading: For this problem set: §4.3, 4.5, 4.6

Problem 8.1
A retired professor comes to his office at any time between 9 AM and 1 PM, with all times in that interval being equally likely, and performs a single task. The duration X of the task is exponentially distributed with parameter $\lambda(y) = 1/(5 - y)$ where y is the length of time between 9 AM and his arrival, measured in hours.

(a) What is the expected time that the professor devotes to his task?

(b) What is the expected time that the professor leaves his office?

(c) The professor has a Ph.D. student who on a given day comes to see the professor at a time that is uniformly distributed between 9 AM and 5 PM. If the student does not find the professor, he leaves and does not return. If he finds the professor, he spends an amount of time that is uniformly distributed between 0 and 1 hour. The professor will spend the same amount of total time on the task regardless of whether he is interupted by the student. What is the expected amount of time that the professor will spend with the student and what is the expected time that he will leave his office? (You do not have to numerically evaluate any integrals which do not have a closed form solution.)

Problem 8.2

(a) Prove the following relation:

$$\mathbb{E}[X] = \mathbb{E}[X|X < \alpha] \cdot \mathbb{P}(X < \alpha) + \mathbb{E}[X|X \geq \alpha] \cdot \mathbb{P}(X \geq \alpha). \tag{1}$$

(b) *Markov Inequality*: Prove that for any non-negative random variable X and for all $\alpha > 0$:

$$\mathbb{P}(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha}.$$

Hint: Part (a) could be useful.

(c) Find an example of a random variable X for which the Markov Inequality is satisfied by equality.

Problem 8.3
Consider n independent tosses of a die with K faces. Suppose that each toss has probability p_i of of resulting in number $i = 1, \ldots, K$. Let X_i be the number of tosses that result in

i. Show that X_1 and X_2 are negatively correlated (i.e., a large number of ones suggests a smaller number of twos).

Problem 8.4

The receiver in an optical communications system uses a photodetector that counts the number of photons that arrive during the communication session. (The time of the communication session is 1 time unit.)

The sender conveys information by either transmitting or *not* transmitting photons to the photodetector. The probability of transmitting is p. If she transmits, the number of photons X that she transmits during the session has a Poisson PMF with mean λ per time unit. If she does not transmit, she generates no photons.

Unfortunately, regardless of whether or not she transmits, there may still be photons arriving at the photodetector because of a phenomenon called shot noise. The number N of photons that arrive because of the shot noise has a Poisson PMF with mean μ. N and X are independent. The total number of photons counted by the photodetector is equal to the sum of the transmitted photons and the photons generated by the shot noise effect.

(a) What is the probability that the sender transmitted if the photodetector counted k photons?

(b) Before you know anything else about a particular communication session, what is the optimal least squares estimate of the number of photons transmitted?

(c) What is the optimal least squares estimate of the number of photons transmitted by the sender if the photodetector counted k photons?

(b) What is the best *linear* predictor for the number of photons transmitted by the sender as a function of k, the number of the detected photons?

Problem 8.5

Consider two random variables X and Y such that $\mathbb{E}[X \mid Y] = 0 = \mathbb{E}[X]$. Are the following statements true or false?

(a) X and Y are uncorrelated.

(b) X and Y are independent.

Justify your answers with an argument, or with a counterexample.