UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: Probability and Random Processes

Discussion Notes: Week 13
Fall 2007

Reading: Berstsekas & Tsitsiklis, §6.3, §6.4, §7.1

Key Stuff to Remember:

- **Markov chains** consist of a set of states and a transition matrix p where p_{ij} gives the probability of transitioning to state j from state i, namely:

$$p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = \mathbb{P}(X_{n+1} = j | X_n = i)$$

- **The Chapman-Kolmogorov Equation for the n-Step Transition Probabilities** is an overly-confusing statement of a fairly obvious recurrence (think of computing the probabilities of all possible paths):

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)p_{kj}, \text{ for } n > 1, \text{ and all } i, j, \; r_{ij}(1) = p_{ij}$$

- **A birth-death Markov Chain** is one in which the states are linearly arranged and transitions can only occur to a neighboring state. As a result, we can solve for the steady-state probabilities with the **local balance equations:**

$$\pi_i = \pi_0 \frac{b_0 b_1 \cdots b_{i-1}}{d_1 d_2 \cdots d_i}$$

- **Some Inequalities on Moments:**

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}, \; \forall a > 0 \qquad \mathbb{P}(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2}, \; \forall c > 0$$

Problem 13.1

A busy professor likes taking each Saturday to sail to his cottage on a nearby island off the coast. The professor is an avid fisherman, and enjoys fishing off of his boat on the way to and from the island, as long as the weather is good. Unfortunately, the weather is good on the way to or from the island with probability p, independently of what the weather was on any past trip (so the weather could be nice on the way to the island, but poor on the way back). Now, if the weather is nice, the professor will take one of his n fishing rods for the trip, but if the weather is bad, he will not bring a fishing rod with him. We want to find the probability that on a given leg of the trip to or from the island the weather will be nice, but the professor will not fish because all his fishing rods are at his other home.

(a) Formulate an appropriate Markov chain to model the above process, keeping the question in mind. The Markov chain should have $n+1$ states. Be sure to specify when a transition occurs. Also explain why it is in fact a Markov chain.

(b) Find the steady state probabilities in the above chain.

(c) What is the probability that on a given trip, the professor sails with clement weather but without a fishing rod?

(d) If the professor owns 4 fishing rods, find p such that the time he wants to, but cannot fish is maximized.

Answers:

(a) We need to know the number of fishing rods on and off the island. It is enough to know the number of fishing rods off the island. Therefore the states of our chain will be the number of fishing rods off the island. We will consider the state of the chain after a round trip to and from the island. This chain clearly has the Markov property, and thus it is easy to see that it is in fact a Markov chain.

(b)

$$\begin{aligned}
p_{ii} &= \begin{cases} (1-p)^2 + p^2 & \text{for } 1 \le i \le n-1 \\ (1-p) & \text{for } i = 0 \\ 1-(1-p)p & \text{for } i = n \end{cases} \\
p_{i,i+1} &= \begin{cases} (1-p)p & \text{for } 1 \le i < n \\ p & \text{for } i = 0 \end{cases} \\
p_{i,i-1} &= \begin{cases} (1-p)p & \text{for } 1 \le i \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Using the fact that this is a birth-death Markov chain, we have:

$$\pi_0 p_{01} = \pi_1 p_{10} \Rightarrow \pi_1 = \frac{\pi_0}{1-p}$$

$$\pi_n = \cdots = \pi_2 = \pi_1 = \frac{\pi_0}{1-p}$$

$$\sum_i \pi_i = \pi_0 (1 + \frac{n}{1-p}) = 1.$$

The steady-state probabilities are

$$\pi_0 = \frac{1-p}{n+1-p}$$

$$\pi_i = \frac{1}{n+1-p}, \text{for all } i > 0.$$

(c) $\pi_n(1-p)p + \pi_0(p) = \frac{2p-2p^2}{n+1-p}$

(d) $p = \frac{10-\sqrt{80}}{2} = 5 - \sqrt{20} = 5 - 2\sqrt{5}$

Problem 13.2

A discrete-time Markov chain is known to be a birth-death process with three states—the probabilities $b_0 = b_1 = d_1 = 0.2$ but d_2 is unspecified. Let X_k denote the state at time $k = 0, 1, \ldots$. For each of the following statements, *either* determine the possible value(s) of probability d_2 *or* explain why no such value exists:

(a) Upon entering state 2, the expected time it takes to first exit state 2 is equal to 100.

(b) The probability $\mathbb{P}(X_2 = 2 \mid X_0 = 2)$ is equal to 0.28.

(c) Given no self-transitions ever occur, the probability $\mathbb{P}(X_4 = 0 \mid X_0 = 0)$ is equal to 0.5.

(d) The steady-state probability associated with state 1 is equal to 0.25.

(e) The long-term expected frequency of deaths is equal to 0.3.

(f) State 2 is an absorbing state.

(g) The mean recurrence time of state 2 is equal to 6.

Answers:

(a) Upon entering state 2, the probability of transitioning out of state 2 is a geometric r.v. with $\rho = d_2$. Therefore, the mean is equal to $\frac{1}{d_2} = 100$. So $d_2 = .01$

(b) $\mathbb{P}(X_2 = 2 \mid X_0 = 2) = \mathbb{P}(X_1 = 1, X_2 = 2 \mid X_0 = 2) + \mathbb{P}(X_1 = 2, X_2 = 2 \mid X_0 = 2)$
$= d_2(.2) + (1 - d_2)^2$
So $d_2(.2) + (1 - d_2)^2 = 0.28$, $d_2^2 - 1.8d_2 + 0.72 = 0$
Solve for d_2: $d_2 = 1.2$ or 0.6. Because d_2 cannot be greater than 1, $d_2 = 0.6$.

(c) Given that there are no self transitions and $X_0 = 0$, $\mathbf{P}(X_3 = 1) = 1$. Since from state 1 we can go to state 0 or 2 with equal probability, the $\mathbf{P}(X_4 = 0|X_0 = 0) = \frac{1}{2}$. This does not depend on d_2. Therefore, d_2 can be any value between 0 and 1.

(d) For $\pi_1 = .25$, using the local balance equations:

$$pi_0(.2) = \pi_1(.2)$$
$$\pi_1(.2) = \pi_2 d_2$$

Thus $\pi_0 = .25, \pi_2 = .5$ and $d_2 = 0.2$.

(e) We can use the long-term expected frequency of deaths along with the local balance equations to solve and get negative probabilities, thus indicating that no such value of d_2 exists.

(f) Since State 2 is an absorbing state, $(1 - d_2)$ has to be 1, $1 - d_2 = 1$. Therefore, $d_2 = 0$.

(g) The mean recurrence time to state 2 is: $1 + p_{20}t_0 + p_{21}t_1 + p_{22}t_2 = 1 + 0 + 15d_2 + 0 = 6$, so $d_2 = \frac{1}{2}$

Problem 13.3

Joe wishes to estimate the true fraction f of smokers in a large population without asking each and every person. He plans to select n people at random and then employ the estimator $F = S/n$, where S denotes the number of people in a size-n sample who are smokers. Joe would like to sample the minimum number of people, but also guarantee an upper bound p on the probability that the estimator F differs from the true value f by a value greater than or equal to d i.e., for a given accuracy d and given confidence p, Joe wishes to select the minimum n such that

$$\mathbb{P}(|F - f| \geq d) \leq p \quad .$$

For $p = 0.05$ and a particular value of d, Joe uses the Chebyshev inequality to conclude that n must be at least 50,000. Determine the new minimum value for n if:

(a) the value of d is reduced to half of its original value.

(b) the probability p is reduced to half of its original value, or $p = 0.025$.

Answers:

(a) $N = 200,000$.

(b) $N = 100,000$.