UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: Probability and Random Processes

Discussion Notes: Week 7 (HINTS)
Fall 2007

Reading: Berstsekas & Tsitsiklis, §3.4, §3.5

Key Stuff to Remember:

- **Continuous Random Variables, PDF and CDF:**

$$\begin{aligned}\mathbb{P}(X \in B) &= \int_B f_X(x)dx \\ F_X(x) &= \mathbb{P}(X \leq x) = \begin{cases} \sum_{k \leq x} p_X(k) & \text{if } X \text{ is discrete.} \\ \int_{-\infty}^{x} f_X(t)dt & \text{if } X \text{ is continuous.} \end{cases}\end{aligned}$$

- **Multiple Continuous Random Variables:**

$$\begin{aligned}P(X,Y) \in B) &= \iint_{(x,y)\in B} f_{X,Y}(x,y)dxdy \\ f_{X,Y}(x,y) &= f_Y(y) f_{X|Y}(x|y) \\ f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy\end{aligned}$$

- **Exponential Random Variables:**

$$f_X(x) = \lambda e^{-\lambda x}, \quad F_X(x) = 1 - e^{-\lambda x}, \quad E(X) = \frac{1}{\lambda}, \quad Var(X) = \frac{1}{\lambda^2}$$

Problem 7.1
One of two wheels of fortune, A and B, is selected by the flip of a fair coin, and the wheel chosen is spun once to determine the experimental value of random variable X. Random variable Y, the reading obtained with wheel A, and random variable W, the reading obtained with wheel B, are described by the PDFs

$$f_Y(y) = \begin{cases} 1, & 0 < y \leq 1; \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad f_W(w) = \begin{cases} 3, & 0 < w \leq \frac{1}{3}; \\ 0, & \text{otherwise.} \end{cases}$$

If we are told the experimental value of X was less than $\frac{1}{4}$, what is the conditional probability that wheel A was the one selected?
Answer:
$\frac{1}{4}$

Problem 7.2
A group of n archers shoot at a target. The distance of each shot from the center of the target is uniformly distributed between 0 to 1, independently of the other shots.

(a) Find the expected distance from the winner's arrow to the center.

(b) Find the expected distance from the loser's arrow to the center (his arrow is the arrow farthest away from the origin).

(c) Find the CDF of the i-th (ranked by distance) player's arrow distance to the center.

Answer:

(a) $\frac{1}{n+1}$ by converting the min expression into the intersection of independent events.

(b) $\frac{n}{n+1}$

(c) Use $\mathbb{P}\left(X_{(i)} \leq x\right) = \mathbb{P}(\text{at least } k \text{ of the } n\ X\text{s are} \leq x)$.

Problem 7.3

Suppose n runners run a race. At one point, all the runners are uniformly distributed on a stretch of one mile that starts at point A and ends at point B. Number the runners from 1 to n, according to the order in which they are at this particular moment. Let X_i denote the distance between point A and the ith runner and let $X_0 = 0$ and $X_{n+1} = 1$.

(a) For $n = 2$, find $\mathbb{P}(X_i > X_{i-1} + t)$ for $i = 1, 2, 3$.

(b) For $n \geq 1$, find $\mathbb{P}(X_i > X_{i-1} + t)$ for $i = 1, 2, \ldots, n+1$.
(Hint: The answer does not depend on i.)

Answer for part (b):

The joint PDF of $(X_1, \ldots, X_n)$ is uniform on the region $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1$, and the integral over the whole region is:

$$\int_0^1 dx_1 \int_{x1}^1 dx_2 \cdots \int_{x_{(n-1)}}^1 \text{const}.dx_n = 1$$

$\mathbb{P}\left(X_i > X_{(i-1)} + t\right)$ is the region:

$$\int_0^{1-t} dx_1 \cdots \int_{x_(i-2)}^{1-t} dx_{(i-1)} \int_{x_i+t}^1 dx_i \cdots \int_{x_{(n-1)}}^1 \text{const}.dx_n$$

The ratio of the two integrals is the result: $(1-t)^n$

Problem 7.4

A car dealer sells two models of cars, Ray and Sprint. A Ray car breaks after time R days, where R is exponentially distributed with parameters λ_R. Similarly, a Sprint car breaks after a random time S that is exponentially distributed with parameters λ_S. R and S are independent random variables.

When broken, a car is brought to the dealer for repair. The cost of fixing a broken Ray is a random variable uniformly distributed over $[100; 300]$ dollars, whereas the cost of fixing a broken Sprint is a random variable that is uniformly distributed over $[200; 400]$ dollars. Different cars have independent costs.

(a) Given that a Ray did not break in the first h days, what is the expected time before the Ray breaks?

(b) What is the probability that a Ray breaks before a Sprint?

(c) On some day, the dealer has to fix N_R cars of the Ray model, where N_R is a Poisson random variable with parameter μ_R. On the same day, the dealer has to fix N_S cars of the Sprint model, where N_S is a Poisson random variable with parameter μ_S. The dealer is interested in estimating the total revenue Z (a random variable) that he will earn from repairs on cars. Compute the moment generating function of Z.

Answer:

(a) $h + \frac{1}{\lambda_R}$

(b) $\frac{\lambda_R}{\lambda_R + \lambda_S}$

(c) $M_Z(s) = e^{\mu_S\left(\frac{e^{300b} - e^{100s}}{s200} - 1\right) + \mu_T\left(\frac{e^{400s} - e^{200s}}{s200} - 1\right)}$