UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: Probability and Random Processes

Discussion Notes: Week 8 (HINTS)
Fall 2007

Reading: Berstsekas & Tsitsiklis, §4.1 – §4.4

Key Stuff to Remember:

- **Transforms as Expectations:**

$$M_X(s) = \mathbb{E}\left[e^{sX}\right] = \begin{cases} \sum_x e^{sx} p_X(x), & X \text{ discrete} \\ \int_{-\infty}^{+\infty} e^{sx} f_X(x)dx, & X \text{ continuous} \end{cases}$$

- **If X and Y are independent** then $M_{X+Y}(s) = M_X(s)M_Y(s)$.
- **Some Transform Pairs:**
  - **Bernoulli**(p): $M_X(s) = 1 - p + pe^s$
  - **Exponential**(λ): $M_X(s) = \frac{\lambda}{\lambda - s}, (s < \lambda)$.
- **Iterated Expectation:** $\mathbb{E}[X] = \mathbb{E}_Y[\mathbb{E}[X|Y]]$.
- **Total Variance:** $\text{var}(X) = \mathbb{E}[\text{var}(X|Y)] + \text{var}(\mathbb{E}[X|Y])$.
- **Random Sums:** for $Y = X_1 + \cdots X_N$ with $\{X_i\}$ i.i.d. and N an RV, we have:
  - $\mathbb{E}[Y] = \mathbb{E}[X_i]\mathbb{E}[N]$
  - $\text{var}(Y) = \text{var}(X_i)\mathbb{E}[N] + (\mathbb{E}[X])^2 \text{var}(N)$
  - $M_Y(s) = M_N(s)|_{e^s = M_X(s)}$

Problem 8.1
Let X, Y, and Z be independent random variables. X is Bernoulli with $p = 1/4$. Y is exponential with parameter 3. Z is Poisson with parameter 5.

(a) Find the transform of $5Z + 1$. *Hint: Write it as an expectation expression and massage the exponential, or check the formula on p. 212*

(b) Find the transform of $X + Y$. *Hint: Since X and Y are independent, the transform of their sum is the product of their transforms.*

(c) Consider the new random variable $U = XY + (1 - X)Z$. Find the transform associated with U. *Hint: Write it as an expectation and apply iterated expectation to fix X*

Problem 8.2
Let L be a discrete random variable whose possible experimental values are all nonnegative integers. We are given

$$M_L(s) = K\frac{16 + 5e^s - 3e^{2s}}{6(2 - e^s)}$$

Determine the numerical values of $E[L]$, $p_L(1)$, and $E[L|L \neq 0]$.
Answers: This question turned out to be kind of lame, so I would *not* suggest studying from it. —Matt

(a) We first need to find the value of K, which is a constant. We know by normalization that the transform evaluated at $s = 0$ must equal 1, so we get $K = 1/3$.

Next, we find the expected value by differentiating with respect to s once and evaluating the result at $s = 0$, giving $\frac{17K}{6} = 17/18$.

(b) Notice that the random variable in question is *nonnegative*. Write the transformation as an expectation and expand out the terms in the sum. If you differentiate this expression with respect to e^s (not s), every term will have an e^s in it except for the one including $p_L(1)$:

$$\begin{aligned} \mathbb{E}\left[e^{sX}\right] &= p_L(0) + p_L(1)e^s + p_L(2)e^{2s} + p_L(3)e^{3s} + \cdots \\ \frac{d\mathbb{E}\left[e^{sX}\right]}{de^s} &= 0 + p_L(1) + 2p_L(2)e^s + 3p_L(3)e^{2s} + \cdots \end{aligned}$$

Therefore, evaluating the transform expression at $e^s = 0$ would give us $p_L(1)$, or $\frac{13}{36}$.

(c) Recall that $p_{L|A}(l|A) = \begin{cases} \frac{p_L(l)}{P(A)} & \text{if } l \in A \\ 0 & \text{otherwise} \end{cases}$. In this case, $A = \{L \neq 0\}$. Therefore,

$$E[L \mid L \neq 0] = \sum_{l=1}^{\infty} l p_{L|A}(l \mid A) = \sum_{l=1}^{\infty} \frac{l p_L(l)}{P(A)} = \frac{E[L]}{P(A)} = \frac{17/18}{P(A)}.$$

(Note that in this case $E[L] = \sum_{l=0}^{\infty} l p_L(l) = \sum_{l=1}^{\infty} l p_L(l)$ since the contribution of the term involving $l = 0$ is 0.)

The probability that L does not equal zero can be found by subtracting the probability that L does equal zero from 1:

$$\begin{aligned} Pr\{L \neq 0\} &= 1 - Pr\{L = 0\} = 1 - [M_L(s)]_{e^s=0} \\ &= 1 - \left(\frac{1}{3}\right)\left(\frac{16}{12}\right) \\ &= 1 - 4/9 = 5/9. \end{aligned}$$

Therefore, $E[L \mid L \neq 0] = \frac{17/18}{5/9} = \boxed{\frac{17}{10}}$.

Problem 8.3
Random variable X is uniformly distributed between -1 and 1. Random variable Y is uniformly distributed between 0 and 2. Find the PDF for $Z = X + Y$ assuming X and Y are independent. *Hint: Use convolution.*

Problem 8.4
Imagine that the number of people that enter a bar in a period of 15 minutes has a Poisson distribution with rate λ. Each person who comes in buys a drink. If there are N types of

drinks, and each person is equally likely to choose any type of drink, independently of what anyone else chooses, find the expected number of different types of drinks the bartender will have to make. *Hint: Let D be the number of types of drinks the bartender makes, and let M be the number of people to enter the bar. Let $X_1, \ldots, X_N$ be the respective indicator variables of each drink. Thus if at least one person orders drink i, then $X_i = 1$, otherwise it equals 0. Also note the relationship between D and $\{X_i\}$. The final answer is $N - Ne^{-\frac{\lambda}{N}}$.*

Problem 8.5

Iwana Passe is taking a quiz with 12 questions. The amount of time she spends answering question i is T_i, and is exponentially distributed with $E[T_i] = \frac{1}{3}$ hour. The amount of time she spends on any particular question is independent of the amount of time she spends on any other question. Once she finishes answering a question, she immediately begins answering the next question.

Let N be the total number of questions she answers **correctly**.

Let X be the total amount of time she spends on questions that she answers **correctly**.

For parts (a) and (b), suppose we know she has probability $\frac{2}{3}$ of getting any particular quiz question correct, independently of her performance on any other quiz question.

(a) Find the expectation and variance of X.

(b) Assuming we know she spent at least $\frac{1}{6}$ of an hour on each question, find the transform of X.

For parts (c) and (d), suppose she has a fixed probability P of getting any particular quiz question correct, independently of her performance on any other quiz question, and with P uniformly distributed between 0 and 1. Assume P is the same value for all questions.

(c) Find the expectation and variance of N.

(d) Assuming there is only one question on the quiz, find the transform of N.

Answers:

(a) $\mathbb{E}[X] = \frac{8}{3}$, $\text{var}(X) = \frac{32}{27}$.

(b) $M_X(s) = \left(\frac{1}{3} + \frac{2}{3}\frac{3e^{\frac{s}{6}}}{(s-3)}\right)^{12}$

(c) $\mathbb{E}[N] = 6$, $\text{var}(N) = 14$ using total variance.

(d) $M_N(s) = 1 - \frac{1}{2} + \frac{1}{2}e^s = \frac{1}{2} + \frac{1}{2}e^s$.