UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: Probablity and Random Processes

Problem Set 8
Fall 2007

Issued: Thursday, October 25, 2007 **Due:** Friday, November 2, 2007

Reading: Bertsekas & Tsitsiklis, §4.4–4.7

Problem 8.1
Let $X_1, X_2, \ldots$ be independent and identically distributed random variables, where each X_i is distributed according to the *logarithmic* PMF with parameter p i.e.,

$$p_X(k) = \frac{(1-p)^k}{k\ln(1/p)}, \quad k = 1, 2, 3, \ldots,$$

where $0 < p < 1$. Discrete random variable N has the Poisson PMF with parameter λ i.e.,

$$p_N(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots,$$

where $\lambda > 0$.

(a) Determine the transform $M_X(s)$ associated with each random variable X_i. *Hint:* You may find the following identity useful:

$$\text{Provided } -1 < a \leq 1, \quad \ln(1+a) = a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \ldots$$

(b) Defining $Y = \sum_{i=1}^{N} X_i$, determine the transform $M_Y(s)$.

Problem 8.2
N male-female couples at a dinner party play the following game. Each member of the couple, writes his/her name on a piece of paper (so there are a total of $2N$ pieces of paper). Then men throw their paper into hat A, while the women throw their paper in hat B. Once all the papers are in, the host draws a piece of paper from each hat, and the two people chosen are dance partners for the rest of the night. The host continues in this fashion until all $2N$ poeple are paired up for night. Let M be the number of couples that are reunited by this game. Find $\mathbb{E}[M]$ and $Var(M)$. *Hint: set up indicator variables and use covariance.*

Problem 8.3
Suppose that the random variables X_1, X_2 have joint density function:

$$f_{X_1,X_2}(x_1, x_2) = \gamma \cdot \exp\left[\frac{-1}{2}(x_1^2 - 6x_1x_2 + 10x_2^2)\right], \text{ for } -\infty < x_1, x_2 < \infty.$$

(a) Find $\mu_{x_1}, \mu_{x_2}, Cov(x_1, x_2)$ and the correlation coefficient ρ.

(b) Find $\mathbb{E}[x_2|x_1 = 3]$.

(c) For $y_1 = ax_1 + bx_2$ and $y_2 = cx_2$ find a, b, c such that $\rho = 0, \sigma_{y_1}^2 = \sigma_{y_2}^2 = 1$.

Problem 8.4

The receiver is an optical communications system uses a photodetector that counts the number of photons that arrive during the communication session. (The time of the communication session is 1 time unit.)

The sender conveys information by either transmitting or not transmitting photons to the photodetector. The probability of transmitting is p. If she transmits, the number of photons X that she transmits during the session has a Poisson PMF with mean λ per time unit. If she does not transmit, she generates no photons.

Unfortunately, regardless of whether or not she transmits, there may still be photons arriving at the photodetector because of a phenomenon called shot noise. The number N of photons that arrive because of the shot noise has a Poisson PMF with mean μ. N and X are independent. The total number of photons counted by the photodetector is equal to the sum of the transmitted photons and the photons generated by the shot noise effect.

(a) What is the probability that the sender transmitted if the photodetector counted k photons?

(b) Before you know anything else about a particular communication session, what is your least squares estimate of the number of photons transmitted?

(c) What is the least squares estimate of the number of photons transmitted by the sender if the photodetector counted k photons?

(d) What is the best *linear* predictor for the number of photons transmitted by the sender as a function of k, the number of the detected photons?

Problem 8.5

Let X and Y be two random variables with positive variances, associated with the same experiment.

(a) Show that ρ, the correlation coefficient of X and Y, equals to -1 if and only if there exists a constant **b** and a *negative* constant **a** such that $Y = aX + b$.

(b) Let $\hat{X}_L$ be the linear least mean squares estimator of X based on Y. Show that

$$\mathbb{E}[(X - \hat{X}_L)Y] = 0.$$

Use this property to show that the correlation of the estimation error $X - \hat{X}_L$ with Y is zero.

(c) Let $\hat{X} = \mathbb{E}[X|Y]$ be the least mean squares estimator of X given Y. Show that

$$\mathbb{E}[(X - \hat{X})h(Y)] = 0,$$

for any function h.

(d) Is it true that the estimation error $X - \mathbb{E}[X|Y]$ is independent of Y?

Problem 8.6

Let random variables X and Y have the bivariate normal PDF

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{\left(x^2 - 2\rho xy + y^2\right)}{2\left(1-\rho^2\right)}\right\}, \quad -\infty < x, y < \infty \qquad ,$$

where ρ denotes the correlation coefficient between X and Y.

(a) Determine the numerical values of $\mathbb{E}[X]$, $\mathrm{var}(X)$, $\mathbb{E}[Y]$ and $\mathrm{var}(Y)$.

(b) Show that X and $Z = (Y - \rho X)/\sqrt{1-\rho^2}$ are independent normal random variables, and determine the numerical values of $\mathbb{E}[Z]$ and $\mathrm{var}(Z)$.

(c) Deduce that

$$\mathbf{P}\left(\{X > 0\} \bigcap \{Y > 0\}\right) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho \quad .$$