UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: Probability and Random Processes

Practice Problems for Midterm #1
Fall 2007

Issued: Thurs, September 27, 2007 **Solutions:** Posted on Tues, October 2, 2007

Reading: Bertsekas & Tsitsiklis, Chapters 1 and 2

Problems for §2.6 and §2.7

Problem 3.1

Let X and Y be independent random variables that take values in the set $\{1, 2, 3\}$. Let $V = 2X + 2Y$, and $W = X - Y$.

(a) Assume that $\mathbb{P}(X = k)$ and $\mathbb{P}(Y = k)$ are positive for any $k \in \{1, 2, 3\}$. Can V and W be independent? Explain. (No calculations needed.)

For the remaining parts of this problem, assume that X and Y are uniformly distributed on $\{1, 2, 3\}$.

(b) Find and plot $p_V(v)$, and compute $\mathbb{E}[V]$ and $\text{var}(V)$.

(c) Find and show in a diagram $p_{V,W}(v, w)$.

(d) Find $\mathbb{E}[V \mid W > 0]$.

(e) Find the conditional variance of W given the event $V = 8$.

Problem 3.2

Joe Lucky plays the lottery on any given week with probability p, independently of whether he played on any other week. Each time he plays, he has a probability q of winning, again independently of everything else. During a fixed time period of n weeks, let X be the number of weeks that he played the lottery and Y the number of weeks that he won.

(a) What is the probability that he played the lottery any particular week, given that he did not win anything that week?

(b) Find the conditional PMF $p_{Y|X}(y \mid x)$.

(c) Find the joint PMF $p_{X,Y}(x, y)$.

(d) Find the marginal PMF $p_Y(y)$. *Hint:* One possibility is to start with the answer in (c), but the algebra can be messy. But if you think intuitively about the procedure that generates Y, you may be able to guess the answer.

(e) Find the conditional PMF $p_{X|Y}(x \mid y)$.

In all parts of this problem, make sure to indicate the range of values to which your PMF formula applies.

Problem 3.3

Suppose you and your friend play a game where each of you throws a 6-sided die, each of your throws being independent. Each time you play the game, if the largest of the two values you obtained from each die is greater than 4, then you win 1 dollar; otherwise, you lose 1 dollar. Suppose that you play the game $n \geq 3$ times, each game being independent of the others.

(a) What is the amount of money you expect to win on the first and last game combined?

(b) How much do you expect to win in your last game given that you lost in the first game?

(c) How much do you expect to have won in your last game given that you won the first game and you won a total of m dollars at the end?

(d) What is the probability that you won both the first and last game given that you won a total of m dollars at the end?

Sample midterm problems

Problem 3.4

Since there is no direct flight from San Diego (S) to New York (N), every time Alice wants to go to the New York, she has to stop in either Chicago (C) or Denver (D). Due to bad weather conditions, both the flights from S to C *and* the flights from C to N have independently a delay of 1 hour with probability p. Similarly, at Denver airport, both incoming and outgoing flights are independently subject to a 2 hour delay with probability q. On any given occasion, Alice chooses randomly between the Chicago or Denver routes with equal probability.

(a) What is the average total delay (across both legs of the overall trip) that she experiences in going from S to N?

(b) Suppose Alice arrives at N with a delay of two hours. What is the probability that she flew through C?

(c) Suppose that Alice wants to maximize the probability that she arrives in New York with a total delay < 2 hours. Under what conditions on p and q is going via Chicago a better choice than going via Denver?

(d) Suppose now that Alice always flies through C. On average, how many trips does she make before experiencing a 2 hour delay?

(e) Suppose now that the flight between S and D is known to be delayed, but Alice still randomly flies either via C or D with equal probability. With what delay should she expect to arrive at N?

Problem 3.5

We transmit a bit of information which is 0 with probability $1-p$ and 1 with p. Because of noise on the channel, each transmitted bit is received correctly with probability $1-\epsilon$.

(a) Suppose we observe a "1" at the output. Find the conditional probability p_1 that the transmitted bit is a "1".

(b) Suppose that we transmit the same information bit n times over the channel. Calculate the probability that the information bit is a "1" given that you have observed n "1"s at the output. What happens when n grows?

(c) For this part of the problem, we suppose that we transmit the symbol "1" a total of n times over the channel. At the output of the channel, we observe the symbol "1" three times in the n received bits, and that we observe a "1" at the n-th transmission. Given these observations, what is the probability that the k-th received bit is a "1"?

(d) Going back to the situation in part (a): some unknown bit is transmitted over the channel, and the received bit is a "1". Suppose in addition that the same information bit is transmitted a second time, and you again receive another "1". We want to find a recursive formula to update p_1 to get p_2, the conditional probability that the transmitted bit is a "1" given that we have observed two "1"s at the output of the channel. Show that the update can be written as

$$p_2 = \frac{(1-\epsilon)p_1}{(1-\epsilon)p_1 + \epsilon(1-p_1)}$$

Problem 3.6

You play the lottery by choosing a set of 6 numbers from $\{1, 2, \ldots, 49\}$ without replacement. Let X be a random variable representing the number of matches between your set and the winning set. (The order of numbers in your set and the winning set does not matter.) You win the grand prize if all 6 numbers match (i.e., if $X = 6$).

(a) What is the probability of winning the grand prize? Compute the PMF p_X of X.

(b) Suppose that before playing the lottery, you (illegally) wiretap the phone of the lottery, and learn that 2 of the winning numbers are between 1 and 20; another 2 are between 21 and 40, and the remaining 2 are between 41 and 49. If you use this information wisely in choosing your six numbers, how does your probability of winning the grand prize improve?

(c) Now suppose *instead* that you determine by illegal wiretapping that the maximum number in the winning sequence is some fixed number R (note that R must be 6 or larger). If you use this information wisely in choosing your 6 numbers, how does your probability of winning the grand prize improve?

(d) Use a counting argument to establish the identity

$$\binom{n}{k} = \sum_{r=k}^{n} \binom{r-1}{k-1}.$$