UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: PROBABILITY AND RANDOM PROCESSES

Discussion 3
Fall 2014

Date: Wednesday, September 17, 2014

Problem 1. The covariance of X and Y is defined as follows.

$$\text{cov}(X, Y) = E((X - E(X))(Y - E(Y)).$$

If cov is zero, we say X and Y are uncorrelated. If cov is positive(negative), we say they are positively(negatively) correlated. What do these mean? If two random variables are positively correlated, it suggests that if X is larger than average, then Y tends to be larger than average.

(a) We roll a dice and denote its outcome as A. Then, we define two random variables as follows.

$$X = \mathbf{1}(A = \text{odd}),\ Y = \mathbf{1}(A \in \{2, 3, 4\}).$$

How are they correlated?

(b) Prove/disprove "Independent random variables are uncorrelated."

(c) Prove/disprove "Uncorrelated random variables are independent."

(d) Prove/disprove "The variance of a sum of uncorrelated random variables is the sum of their variances."

Problem 2. Density of Function of Random Variables (**Reading: A.6 in W**). Assume that $\mathbf{Y} = g(\mathbf{X})$ where $\mathbf{X}$ has density $f_{\mathbf{X}}$ in $\mathcal{R}^n$ and $g : \mathcal{R}^n \to \mathcal{R}^n$ is differentiable. Then,

$$f_{\mathbf{Y}}(\mathbf{y}) = \sum_i \frac{1}{|J(\mathbf{x}_i)|} f_{\mathbf{X}}(\mathbf{x}_i)$$

where the sum is over all the $\mathbf{x}_i$ such that $g(\mathbf{x}_i) = \mathbf{y}$ and $J_{i,j}(\mathbf{x}) = \frac{\partial}{\partial x_j} g_i(\mathbf{x})$. The matrix J is called the Jacobian of the function g.

(a) A random variable X is uniformly distributed in $[0, 1]$. Find the distribution of $Y = 3X + 15$.

(b) Let $\mathbf{X} = (X_1, X_2)$ a random vector, where the X_i are uniformly distributed in $[0, 1]$, and independent of each other. Find the distribution of $\mathbf{Y} = (X_1 + 2X_2 + 3, 3X_1 + X_2 + 5)$.

(c) A random variable X is uniformly distributed in $[-1, 1]$. Find the distribution of $Y = X^2$.

(d) Approximate $g(\mathbf{x} + \mathrm{d}\mathbf{x})$ around a given point $\mathbf{x}$, and express it using the Jacobian matrix defined above.

(e) Try to understand the formula given in the beginning of the discussion by collecting all lessons from example b), c), and d).

(f) Let $\mathbf{X} = (X_1, X_2)$ a random vector, where the X_i are uniformly distributed in $[0, 1]$, and independent of each other. Find the distribution of $\mathbf{Y} = (X_1^2 + X_2^2, 2X_1X_2)$.

Problem 3. Let $X_1, X_2, \ldots, X_n$ be i.i.d. continuous random variables distributed uniformly between 0 and 1.

(a) Find $E(X_1)$.

(b) Find $var(X_1)$.

(c) Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the ordered random variables such that $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$. Find the marginal distributions of $X_{(1)}$ and $X_{(n)}$.

(d) Find the expected value of $\min_i X_i = X_{(1)}$ and $\max_i X_i = X_{(n)}$. Can you do it without any calculations?

(e) What is the joint distribution of $(X_{(1)}, X_{(2)}, \ldots, X_{(n)})$? Can you guess the answer without any calculations?