EECS 126 Probability and Random Processes
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(Practice Version) Midterm Exam 1

Last name	First name	SID

Rules.

- DO NOT open the exam until instructed to do so.
- Note that the test has 100 points. The maximum possible score is 100.
- You have 10 minutes to read this exam without writing anything.
- You have 90 minutes to work on the problems.
- Box your final answers.
- Partial credit will not be given to answers that have no proper reasoning.
- **Remember to write your name and SID on the top left corner of every sheet of paper.**
- **Do not write on the reverse sides of the pages.**
- All eletronic devices must be turned off. Textbooks, computers, calculators, etc. are prohibited.
- No form of collaboration between students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- **You must include explanations to receive credit.**

Problem	Part	Max	Points	Problem	Part	Max	Points
1	(a)	6		2		20	
	(b)	6		3		25	
	(c)	6		4		25	
	(d)	6		5		20	
	(e)	6					
		30					
Total						100	

Cheat sheet (this will be provided in the actual midterm.)

1. Discrete Random Variables

1) Geometric with parameter $p \in [0,1]$:
$P(X = n) = (1-p)^{n-1}p,\ n \geq 1$
$E[X] = 1/p,\ \mathrm{var}(X) = (1-p)p^{-2}$

2) Binomial with parameters N and p:
$P(X = n) = \binom{N}{n} p^n (1-p)^{N-n},\ n = 0, \ldots, N,$ where $\binom{N}{n} = \frac{N!}{(N-n)!n!}$
$E[X] = Np,\ \mathrm{var}(X) = Np(1-p)$

3) Poission with parameter λ:
$P(X = n) = \frac{\lambda^n}{n!} e^{-\lambda},\ n \geq 0$
$E[X] = \lambda,\ \mathrm{var}(X) = \lambda$

2. Continuous Random Variables

1) Uniformly distributed in $[a,b]$, for some $a < b$:
$f_X(x) = \frac{1}{b-a} 1\{a \leq x \leq b\}$
$E[X] = \frac{a+b}{2},\ \mathrm{var}(X) = \frac{(b-a)^2}{12}$

2) Exponentially distributed with rate $\lambda > 0$:
$f_X(x) = \lambda e^{-\lambda x} 1\{x \geq 0\}$
$E[X] = \lambda^{-1},\ \mathrm{var}(X) = \lambda^{-2}$

3) Gaussian, or normal, with mean μ and variance σ^2:
$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}$
$E[X] = \mu,\ \mathrm{var} = \sigma^2$

Problem 1. **Short questions: 6 points each.**

(a) Consider a well-shuffled 52-card deck. We draw the top 10 cards of the deck. What is the probability that the 10 cards include exactly 3 queens, or exactly 4 kings, or both?

(b) Suppose that you put m balls randomly in n boxes. Each box can hold an arbitrarily large number of balls. What is the expected number of empty boxes?

(c) Let X and Y be independent and identically distributed random variables with cdf F. Find the CDF of $Z = \max(X, Y)$ and the CDF of $W = \min(X, Y)$.

(d) Find an example of two random variables that are uncorrelated ($cov(X, Y) = 0$), but they are not independent.

(e) Assume that the two random variables X and Y are such that $E[X|Y] = Y$ and $E[Y|X] = X$. Show that $P(X = Y) = 1$. [Hint: Show that $E[(X - Y)^2] = 0$.]

Problem 2. (20 pts) Consider a stick of length 1. You break the stick at a random point, and then break the piece that contains the left end of the stick randomly. What is the probability that the 3 pieces form a triangle?

Problem 3. (25 pts) Consider a bipartite graph with K left nodes and M right nodes as shown in Figure 1. Each right node is connected to a left node with probability p independently. Let D_r be the random variable denoting the degree of a randomly selected right node, and D_l be the random variable denoting the degree of a randomly selected left node.

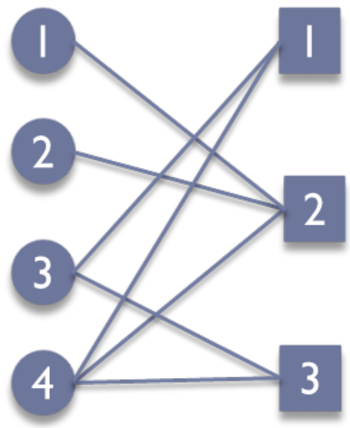

Figure 1: One realization of the bipartite graph.

(a) Find the distribution of D_r and D_l. Find their expectations.

(b) Suppose that you pick a random edge in the graph. What is the distribution of the degree of the left node connected to this edge? What is the distribution of the degree of the right node connected to this edge?

(c) We call a right node with degree 2 a *doubleton*. For example right node 1 is a doubleton. What is the average number of doubletons in the random bipartite graph model?

(d) Find the average number of left nodes that are connected to at least one doubleton.

Problem 4. (25 pts) Jim throws a dart at a circle of radius r that is equally likely to hit any point in the circle. Let X be the distance of Jim's hit from the center.

(a) Find the pdf, the mean, and the variance of X.

(b) The target has an inner circle of radius t. If $X \leq t$, Jim gets a score of $S = 1/X$. Otherwise his score is 0. Find the CDF of S. Is S a continuous random variable?

Problem 5. (20 pts) You are given two biased coins, one of which has the probability 0.55 of coming up Heads, and the other of which has a probability of 0.2 of coming up Heads. However, you don't know which is which.

(a) You randomly pick a coin and flip it. If it comes up Heads, you win $\$1$. If it comes up Tails, you lose $\$1$. What is the probability of winning? Would you play this game?

(b) Now suppose you get to pick a coin randomly and flip it for free. Then, you get to choose a coin for your second flip. Again, you win or lose $\$1$ according to whether the second flip is Heads or Tails. Given that you see Heads on the first flip, which coin would you choose for the second flip? With this chosen coin, would you bet $\$1$ on the second flip?

(c) Now suppose you get to randomly pick a coin and flip it for free n times $(n \geq 1)$, and then pick a coin for your $(n+1)$-th flip. What is your chance of winning?

END OF THE EXAM.

Please check whether you have written your name and SID on every page.