EECS 126 Probability and Random Processes
Kannan Ramchandran

University of California, Berkeley: Fall 2014
November 7, 2014

(Practice Version) Midterm Exam 2

Last name	First name	SID

Rules.

- DO NOT open the exam until instructed to do so.
- Note that the test has 105 points. The maximum possible score is 100.
- You have 10 minutes to read this exam without writing anything.
- You have 90 minutes to work on the problems.
- Box your final answers.
- Partial credit will not be given to answers that have no proper reasoning.
- **Remember to write your name and SID on the top left corner of every sheet of paper.**
- **Do not write on the reverse sides of the pages.**
- All eletronic devices must be turned off. Textbooks, computers, calculators, etc. are prohibited.
- No form of collaboration between students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- **You must include explanations to receive credit.**

Problem	Part	Max	Points	Problem	Part	Max	Points
1	(a)	8		2		15	
	(b)	8		3		15	
	(c)	8		4		20	
	(d)	8		5		15	
	(e)	8					
		40					
Total						105	

Cheat sheet

1. Discrete Random Variables

1) Geometric with parameter $p \in [0,1]$:
$P(X=n) = (1-p)^{n-1}p,\ n \geq 1$
$E[X] = 1/p,\ \text{var}(X) = (1-p)p^{-2}$

2) Binomial with parameters N and p:
$P(X=n) = \binom{N}{n} p^n (1-p)^{N-n},\ n = 0, \ldots, N,$ where $\binom{N}{n} = \frac{N!}{(N-n)!n!}$
$E[X] = Np,\ \text{var}(X) = Np(1-p)$

3) Poission with parameter λ:
$P(X=n) = \frac{\lambda^n}{n!}e^{-\lambda},\ n \geq 0$
$E[X] = \lambda,\ \text{var}(X) = \lambda$

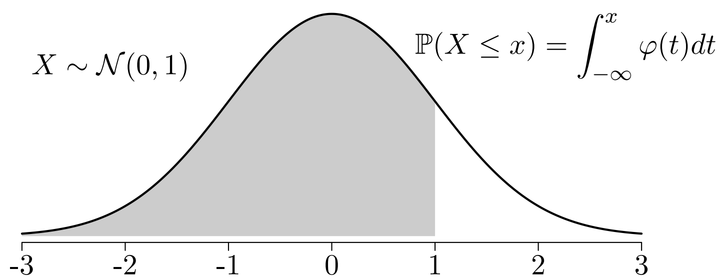
2. Continuous Random Variables

1) Uniformly distributed in $[a,b]$, for some $a < b$:
$f_X(x) = \frac{1}{b-a} 1\{a \leq x \leq b\}$
$E[X] = \frac{a+b}{2},\ \text{var}(X) = \frac{(b-a)^2}{12}$

2) Exponentially distributed with rate $\lambda > 0$:
$f_X(x) = \lambda e^{-\lambda x} 1\{x \geq 0\}$
$E[X] = \lambda^{-1},\ \text{var}(X) = \lambda^{-2}$

3) Gaussian, or normal, with mean μ and variance σ^2:
$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}$
$E[X] = \mu,\ \text{var} = \sigma^2$

3. Normal Distribution Table



	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Problem 1. Short questions.

(a) You roll a die successively. You stop rolling the die when the sum of the last two numbers is 10. Find the average number of times that you roll the die.

(b) Let $\{N_t, t \geq 0\}$ be a Poisson process with rate λ. Note that N_t is the number of arrivals up to time t. Find the maximum-likelihood estimate of λ given N_1 and N_2.

(c) Consider two hypothesis: $X \in \{0, 1\}$. You observe Y that is an exponential random variable with rate $X + 1$. Solve a hypothesis testing problem such that the probability of false alarm is less than or equal β; that is, $\Pr(\hat{X} = 1 | X = 0) \leq \beta$.

(d) Consider a coin that has $\Pr(Heads) = p$. You flip the coin 4 times, and observe the sequence H, T, T, H. What it the maximum-likelihood estimator of p?

(e) Let X be an exponential random variable with rate 1. Use Chebychev's inequality to get a bound on $\Pr(X > a)$.

Problem 2. Consider n light bulbs that have independent lifetimes exponentially distributed with mean 1.

(a) What is the average time until the last bulb dies?

(b) Assume that the janitor replaces a burned out bulb after an exponentially distributed time with mean 0.1. What is the average time until all the bulbs are out? [You just need to write down the equations, and do not need to solve it.]

Problem 3. Consider a discrete Markov chain with the following transition probability matrix:

$$\begin{pmatrix} 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

(a) Is the Markov chain aperiodic? Prove it.

(b) Find the invariant distribution.

(c) Find $E[T_2|X_0 = 1]$.

(d) Construct an infinite periodic Markov chain with period 3.

Problem 4. Let $\{N_t,\ t \geq 0\}$ be a Poisson process with rate λ. To model the bursty arrivals of packets in communication networks, we consider the following model. At each jump time T_n of the process, a random number X_n of packets arrive at a switch. The random variables $\{X_n,\ n \geq 1\}$ are i.i.d. with mean μ and variance σ^2. Let A_t be the number of packets that arrive by time t at the switch, for $t \geq 0$.

(a) Find $E[A_t]$ and $var(A_t)$.

(b) What does $\frac{A_t}{t}$ approach to as $t \to \infty$? Show your work.

(c) What is the approximate distribution of $\frac{A_t - \mu\lambda t}{\sqrt{t}}$? Find x such that $P(A_{100} > x) \simeq 0.05$. Show your work.

Problem 5. You observe n i.i.d. samples $X_1, X_2, \ldots, X_n$ from the distribution $f_{X|\theta}(x) = \frac{1}{2}e^{-|x-\theta|}$, where $\theta \in \mathbb{R}$ is the parameter to estimate. Find $MLE[\theta|X_1, X_2, \ldots, X_n]$.

END OF THE EXAM.

Please check whether you have written your name and SID on every page.