

Problem Set 7

Fall 2014

Issued: Thursday, October 16, 2014

Due: Thursday, October 23, 2014

Problem 1. We flip a fair coin repeatedly until we get either the pattern HHH or HTH . What is the average number of coin flips?

Problem 2. Each morning, as you pull out of your driveway, you would like to make a U-turn rather than drive around the block. Unfortunately, U-turns are illegal and police cars drive by according to a Poisson process with rate λ . You decide to make a U-turn one you see that the road has been clear of police cars for τ units of time. Let N be the number of police cars you see before you make a U-turn.

- (a) Find $E[N]$.
- (b) Find the conditional expectation of the time elapsed between police cars $n - 1$ and n , given that $N \geq n$.
- (c) Find the expected time that you wait until you make a U-turn.

Problem 3. A continuous-time queue has Poisson arrivals with rate λ , and it is equipped with infinitely many servers. The servers can work in parallel on multiple customers, but they are non-cooperative in the sense that a single customer can only be served by one server. Thus, when there are k customers in the queue, k servers are active. Suppose that the service time of each customer is exponentially distributed with rate μ and they are i.i.d.

- (a) Argue that the queue-length is a Markov chain. Draw the transition diagram of the Markov chain.
- (b) Prove that for all finite values of λ and μ the Markov chain is positive recurrent and find the invariant distribution.

Problem 4. Let $\{N_t, t \geq 0\}$ be a Poisson process with rate λ and define $X_t = X_0(-1)^{N_t}$ where $X_0 \in \{0, 1\}$ is a random variable independent of N_t .

- (a) Does the process X_t have independent increments?
- (b) Calculate $\Pr(X_t = 1)$ if $\Pr(X_0 = 1) = p$.

- (c) Assume that $p = 0.5$. Calculate $E(X_{t+s}X_s)$ for $x, t \geq 0$.

Problem 5. A queue has Poisson arrivals with rate λ . It has two servers that work in parallel. Where there are at least two customers in the queue, two are being served. When there is only one customer, only one server is active. The service times are i.i.d. $Exp(\mu)$.

- (a) Argue that the queue length is a Markov Chain.
 (b) Draw the state transition diagram.
 (c) Find the minimum value of μ so that the queue is positive recurrent and solve the balance equations.

Problem 6. Empty taxis pass by a street corner at a Poisson rate of two per minute and pick up a passenger if one is waiting there. Passengers arrive at the street corner at a Poisson rate of one per minute and wait for a taxi only if there are less than four persons waiting; otherwise they leave and never return. John arrives at the street corner at a given time. Find his expected waiting time, given that he joins the queue. Assume that the process is in steady state.

Problem 7. Apple announces a surprise one-day promotion for brand-new iPhone 6: the price of a new phone will be 0 dollar at time 0, and it will increase by 5 dollars per minute! As soon as you wake up and check Twitter feeds(Huffman-coded), you run all the way to the nearest Apple store. Since there are too many Apple fanboys in your city, there are already too many customers waiting in line for service. There are 2 queues: the first queue has i customers including the one being served, and the second queue has j customers as depicted in Figure 1.

Assume that service time for each customer is exponentially distributed with a mean of 10 minutes.

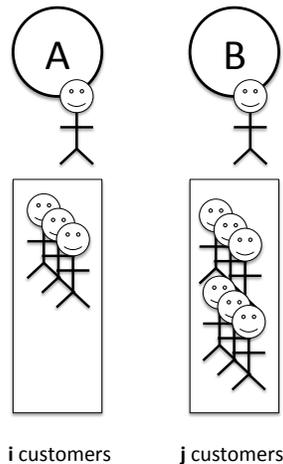


Figure 1: Apple Store on the 4th street, Berkeley

- (a) Assume $i = j = 5$. You randomly choose one of the two queues and wait for service. What will be the expected price of the iPhone6 when your service?
- (b) Assume $i = 10$, $j = 11$. You need to choose one of the two queues. Which queue would you choose? You don't need to rigorously justify your choice but briefly explain why. What will be the expected price of the iPhone6 when you start getting served?
- (c) Assume $i = 10$, $j = 11$. Your younger brother suggests the following: he can wait for you in another queue so that if he reaches the server earlier than you, you can move to the other queue! He wants 50 dollars for helping you with this plan. Is it smart for you to hire him? In other words, would you expect a lower price for your iPhone6 if you hired him?
- (d) Assume $i = 10$, $j = 11$, and you hired your brother. What is the probability that your brother reaches the server earlier than you?

Mini-Lab. Submit your mini-project report with the mini-lab #6 ipynb file.