UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: PROBABILITY AND RANDOM PROCESSES

Discussion 4
Fall 2015

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Transform (Moment Generating Function)

Definition

The transform (moment generating function) of a random variable is defined as

$$M_X(s) = E[e^{sX}].$$

For a random vector $X = (X_1, \ldots, X_n)^T$, the transform of X is

$$M_X(s) = E[e^{s^T X}].$$

Properties

(a) $M_X(0) = 1$.

(b) $M_X(s) = \sum_{k=0}^{\infty} \frac{s^k m_k}{k!}$, where $m_k = E[X^k]$.

(c) $m_k = \frac{d^k M_X(s)}{ds^k}|_{s=0}$.

(d) If two distributions have the same transform, then they are identical at almost all points. That is, if for all values of s,

$$M_X(s) = M_Y(s),$$

then

$$F_X(x) = F_Y(x),$$

(e) The transform of a linear combination of independent random variables is the product of their transforms. More specifically, if $X_1, \ldots, X_n$ are independent, and $Y = \sum_{i=1}^{n} a_i X_i$, then

$$M_Y(s) = \prod_{i=1}^{n} M_{X_i}(a_i s).$$

Transforms of common distributions

(a) Poisson(λ): $\exp\{\lambda(e^s - 1)\}$.

(b) Gaussian $\mathcal{N}(\mu, \sigma^2)$: $e^{s\mu + \frac{1}{2}\sigma^2 s^2}$.

(c) Exponential(λ): $\frac{\lambda}{\lambda - s}$ ($s < \lambda$).

(d) Find transforms of other distributions online.

Examples

Problem 1. Show that if X and Y are independent Poisson random variables with means λ_1 and λ_2, then $X + Y$ is Poisson distributed with mean $\lambda_1 + \lambda_2$.

Problem 2. Consider random variable Z with transform

$$M_Z(s) = \frac{a - 3s}{s^2 - 6s + 8}.$$

(a) Find the numerical value for the parameter a.

(b) Find $P(Z \geq 0.5)$.

(c) Find $E[Z]$ by using the probability distribution of Z.

(d) Find $E[Z]$ by using the transform of Z and without explicity using the probability distribution of Z.

(e) Find $\text{var}(Z)$ by using the probability distribution of Z.

(f) Find $\text{var}(Z)$ by using the transform of Z and without explicity using the probability distribution of Z.