UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 2

Fall 2015

Issued: Thursday, September 03, 2015 **Due:** 9am, Thursday, September 10, 2015

Problem 1. Consider a binary tree with n levels as shown in Figure 1. Suppose that each link in this tree works with probability p, and is defective with probability $1-p$, independently of other links. Find the probability that there is a working path from the root node (R) to a leaf. You only need to find a recursive formula for this probability.

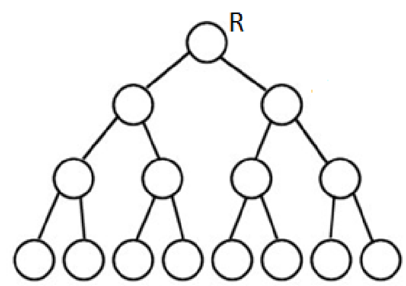

Figure 1: reliability graph for a binary tree with $n = 3$.

Problem 2. Consider an urn contains b blue and r red balls. In each experiment, we randomly pick a ball from the urn, record its color, and put the ball back to the urn. We stop the experiments when red balls have appeared k times. Let X be the random variable denoting the number of experiments we have done before stopping.

(a) Find the distribution (PMF) of X.

(b) Find the expected value of X.

(c) Find the variance of X.

Problem 3. Suppose a random variable X can have only non-negative integer values.

(a) Show that

$$\mathbb{E}[X] = \sum_{x=0}^{\infty} \Pr(X > x).$$

(b) Show that

$$\sum_{x=0}^{\infty} x\Pr(X > x) = \frac{1}{2}\left(\mathbb{E}\left[X^2\right] - \mathbb{E}[X]\right).$$

Problem 4. Suppose there is a 0-1 Bernoulli sequence $X^n = (X_1, X_2, \ldots, X_n)$, where X_i's are i.i.d. Bernoulli random variables with $\Pr(X_i = 1) = p$. We define the *runs* of X^n as follows:
A subsequence $(X_i, X_{i+1}, \ldots, X_j)$ of X^n is called a *run* if $X_i = X_{i+1} = \cdots = X_j$, $X_{i-1} \neq X_i$, and $X_{j+1} \neq X_j$. (Note that if $i = 1$, we do not need $X_{i-1} \neq X_i$, and if $j = n$, we do not need $X_{j+1} \neq X_j$.)
For example, there are 6 runs of $X^n = (0011001110001)$, i.e., $(00), (11), (00), (111), (000)$, and (1). What is the expected number of runs of the 0-1 Bernoulli sequence that we mentioned above?
Hint: Think about transitions from 0 to 1 and from 1 to 0.

Problem 5. There is a set of 52 pokers. Each time we randomly pick one card without putting it back. What is the expected number of cards that we need to pick before we see the first King.
Hint: There are 4 Kings and 48 other cards. For a card that is not a King, think about the probability that it is picked before the first King.

Problem 6. Consider a balls-and-bins model with K balls and M bins as shown in Figure 2. For each pair of ball and bin (e.g. the ith ball and the jth bin), the ball is thrown into the bin with probability p, independently from other pairs. The balls-and-bins model can also be viewed as a random graph. As we can see, the degree of the bins (number of balls thrown in a bin) is a random variable.

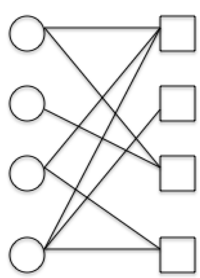

Figure 2: Balls-and-bins model.

(a) Find the distribution of the degree of a bin.

(b) What is the expected degree of a bin?

(c) Now suppose you pick a random edge in the graph. What is the distribution of the degree of the right node connected to this edge? Is it the same as part (a)?

(d) We call a bin with degree 1 a *singleton*. What is the average number of singletons in a random balls-and-bins model?

(e) Find the average number of balls that are connected to at least one singleton.