UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 6
Fall 2015

Issued: Thursday, October 8, 2015 **Due:** 9:00am, Thursday, October 15, 2015

Problem 1. A discrete-time Markov chain with seven states has the following transition probabilities:

$$p_{ij} = \begin{cases} 0.5, & (i,j) = (3,2),(3,4),(5,6) \text{ and } (5,7) \\ 1, & (i,j) = (1,3),(2,1),(4,5),(6,7) \text{ and } (7,5) \\ 0, & \text{otherwise} \end{cases}.$$

In the questions below, let X_k be the state of the Markov chain at time k.

(a) Give a pictorial representation of the discrete-time Markov chain.

(b) For what values of n is $\Pr(X_n = 5 \mid X_0 = 1) > 0$?

(c) What is the set of states $A(i)$ that are accessible from state i, for each $i = 1, 2, \dots, 7$? Is the Markov chain irreducible?

(d) Identify which states are transient and which states are recurrent. For each recurrent state, state whether it is periodic (and give the period) or aperiodic.

(e) If $X_0 = 1$, what is the expected time for the Markov chain to reach state 7 for the first time?

Problem 2. Consider the Markov chain with state X_n, $n \geq 0$, shown in Figure 1, where $\alpha, \beta \in (0, 1)$.

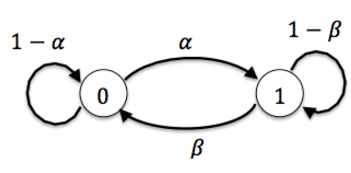

Figure 1: Markov chain for Problem 2

(a) Find the probability transition matrix P and the invariant distribution π of the Markov chain.

(b) Find two real numbers λ_1 and λ_2 such that there exists two non-zero vectors u_1 and u_2 such that $Pu_i = \lambda_i u_i$ for $i = 1, 2$. Further, show that P can be written as $P = U\Lambda U^{-1}$, where U and Λ are 2×2 matrices and Λ is a diagonal matrix.

Hint: This is called the eigendecomposition of a matrix.

(c) Find P^n in terms of U and Λ.

(d) Assume that $X_0 = 0$. Use the result in part (c) to compute the PMF of X_n for all $n \geq 0$. Verify that it converges to the invariant distribution.

Problem 3. (a) Find the steady-state probabilities $\pi_0, \ldots, \pi_{k-1}$ for the Markov chain in Figure 2. Express your answer in terms of the ratio $\rho = p/q$, where $q = 1 - p$. Pay particular attention to the special case $\rho = 1$.

(b) Find the limit of π_0 as k approaches infinity; give separate answers for $\rho < 1$, $\rho = 1$, and $\rho > 1$. Find limiting values of π_{k-1} for the same cases.

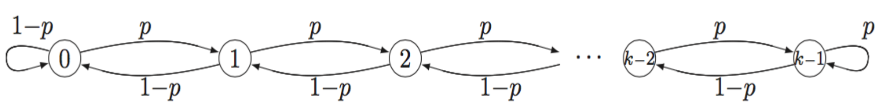

Figure 2: Markov chain for Problem 3

Problem 4. Let $\{X_n, n \geq 0\}$ be a Markov chain with two states, -1 and 1, and transition probabilities $P(-1, 1) = P(1, -1) = a$ for $a \in (0, 1)$. Define,

$$Y_n = X_0 + X_1 + \ldots + X_n.$$

Is $\{Y_n, n \geq 0\}$ a Markov chain? Prove or disprove.

Problem 5. Alice is rolling a fair four-sided dice with labels 1, 2, 3, and 4, and she records all the outcomes of the rolls. What is the expected time that she needs to roll the dice in order to see the four consecutive outcomes which are either $1 \to 2 \to 3 \to 4$ or $4 \to 3 \to 2 \to 1$?

Problem 6. Consider two independent Poisson processes with rates λ_1 and λ_2. Those processes measure the number of customers arriving in store 1 and 2.

(a) What is the probability that a customer arrives in store 1 before any arrives in store 2?

(b) What is the probability that in the first hour exactly 6 customers arrive at the two stores? (The total for both is 6)

(c) Given exactly 6 have arrived at the two stores, what is the probability all 6 went to store 1?

Problem 7. You have a database of an infinite number of movies. Each movie has a rating that is uniformly distributed in $\{0, 0.5, 1, 1.5, 2, 2.5, 3, 3, 3.5, 4, 4.5, 5\}$ independent of all other movies. You want to find two movies such that the sum of their ratings is greater than 7.5 (7.5 is not included).

(a) A Stanford student chooses two movies each time and calculates the sum of their ratings. If it is less than or equal to 7.5, the student throws away these two movies and chooses two other movies. The student stops when he/she finds two movies such that the sum of their ratings is greater than 7.5. What is the expected number of movies that this student needs to choose from the database?

(b) A Berkeley student chooses movies from the database one by one and keeps the movie with the highest rating. The student stops when he/she finds that the sum of the ratings of the last movie that he/she has chosen and the movie with the highest rating among all the previous movies is greater than 7.5. What is the expected number of movies that the student will have to choose?

Now consider the ratings of movies can be real numbers and assume that the ratings are i.i.d. uniformly distributed in $[0, 5]$.

(c) What is the expected number of movies that the Stanford student will have to choose in order to find two movies such that the sum of their ratings is greater than 7.5?

(d) (Optional) What is the expected number of movies that the Berkeley student will have to choose in order to find two movies such that the sum of their ratings is greater than 7.5?