UC Berkeley
Department of Electrical Engineering and Computer Sciences

ELECTRICAL ENGINEERING 126: PROBABILITY AND RANDOM PROCESSES

Discussion 1
Fall 2017

---

1. **Deriving Facts from the Axioms**

   (a) Let $n \in \mathbb{Z}_{>0}$ and $A_1, \dots, A_n$ be any events. Prove the **union bound**: $\mathbb{P}(\bigcup_{i=1}^{n} A_i) \le \sum_{i=1}^{n} \mathbb{P}(A_i)$.

   (b) Let $A_1 \subseteq A_2 \subseteq \cdots$ be a sequence of increasing events. Prove that $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(\bigcup_{i=1}^{\infty} A_i)$. [This can be viewed as a **continuity** property for probability measures.]

   (c) Let $A_1, A_2, \dots$ be a sequence of events. Prove that the union bound holds for countably many events: $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

2. **Balls & Bins**

   Let $n \in \mathbb{Z}_{>1}$. You throw n balls, one after the other, into n bins, so that each ball lands in one of the bins uniformly at random. What is an appropriate sample space to model this scenario? What is the probability that exactly one bin is empty?

3. **Monty Hall Mixed Strategies**

   We showed that in the Monty Hall problem, it is better to switch than to stay. Now we will consider mixed strategies: that is, you choose to switch with probability $p \in [0, 1]$. What is the optimal value of p?

4. **Borel-Cantelli Lemma**

   Prove the **Borel-Cantelli Lemma**: If $A_1, A_2, \dots$ is a sequence of events with $\sum_{i=1}^{\infty} \mathbb{P}(A_i) < \infty$, then

   $$\mathbb{P}(\text{infinitely many of } A_1, A_2, \dots \text{ occur}) = 0.$$