UC Berkeley
Department of Electrical Engineering and Computer Sciences

Electrical Engineering 126: Probability and Random Processes

Discussion 7
Fall 2017

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1. **Backwards Markov Property**

   Let $(X_n, n \in \mathbb{N})$ be a discrete-time Markov chain with state space $\mathcal{S}$. Show that for every $m, k \in \mathbb{N}$, $m \geq 1$, we have

   $$\mathbb{P}(X_k = i_0 \mid X_{k+1} = i_1, \dots, X_{k+m} = i_m) = \mathbb{P}(X_k = i_0 \mid X_{k+1} = i_1)$$

   for all states $i_0, i_1, \dots, i_m$.

2. **Seven-State Chain**

   A discrete-time Markov chain with seven states has the following transition probabilities:

   $$p_{i,j} = \begin{cases} 0.5, & (i,j) = (3,2), (3,4), (5,6), \text{ and } (5,7) \\ 1, & (i,j) = (1,3), (2,1), (4,5), (6,7), \text{ and } (7,5) \\ 0, & \text{otherwise} \end{cases}$$

   In the questions below, let X_k be the state of the Markov chain at time k, for each $k \in \mathbb{N}$.

   (a) Give a pictorial representation of the discrete-time Markov chain.

   (b) For what values of n is $\mathbb{P}(X_n = 5 \mid X_0 = 1) > 0$?

   (c) What is the set of states $A(i)$ that are accessible from state i, for each $i = 1, 2, \dots, 7$?

   (d) If $X_0 = 1$, what is the expected time for the Markov chain to reach state 7 for the first time?

3. **Two-State Chain with Linear Algebra**

   Consider the Markov chain $(X_n,\ n \in \mathbb{N})$, shown in Figure 1, where $\alpha, \beta \in (0, 1)$.

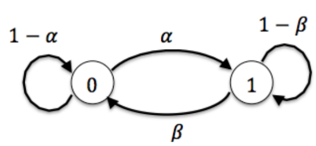

Figure 1: Markov chain for Problem 3.

(a) Find the probability transition matrix P.

(b) Find two real numbers λ_1 and λ_2 such that there exists two non-zero vectors u_1 and u_2 such that $Pu_i = \lambda_i u_i$ for $i = 1, 2$. Further, show that P can be written as $P = U\Lambda U^{-1}$, where U and Λ are 2×2 matrices and Λ is a diagonal matrix.

Hint: This is called the eigendecomposition of a matrix.

(c) Find P^n in terms of U and Λ for each $n \in \mathbb{N}$.

(d) Assume that $X_0 = 0$. Use the result in part (c) to compute the PMF of X_n for all $n \in \mathbb{N}$.