UC Berkeley
Department of Electrical Engineering and Computer Sciences

Electrical Engineering 126: Probability and Random Processes

Problem Set 2

Fall 2017

Issued: September 7, 2017 **Due:** 9 AM, Thursday, September 14, 2017

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1. **Packet Routing**

   Consider a system with n inputs and n outputs. At each input, a packet appears independently with probability p. If a packet appears, it is destined for one of the n outputs uniformly randomly, independently of the other packets.

   (a) Let X denote the number of packets destined for the first output. What is the distribution of X?

   (b) What is the probability of a collision, that is, more than one packet heading to the same output?

2. **Numbered Balls**

   A bin contains balls numbered $1, 2, \ldots, n$. You reach in and select k balls at random. Let T be the sum of the numbers on the balls you picked.

   (a) Say $k = 1$, what is $\mathbb{E}[T]$?

   (b) Find $\mathbb{E}[T]$ for general values of k.

   (c) What is $\text{var}(T)$?

3. **Poisson Properties**

   (a) Suppose X and Y are independent Poisson random variables with mean λ and μ respectively. Prove that $X + Y$ has the Poisson distribution with mean $\lambda + \mu$. (This is known as **Poisson merging**.)

   (b) Suppose X is an exponential random variable with mean $1/\lambda$, that is, X is a continuous random variable with density $f_X(x) = \lambda \exp(-\lambda x)$ for $x > 0$. Show that

   $$\mathbb{E}(X^k) = \frac{k!}{\lambda^k}.$$

4. **Indicators & Markov's Inequality**

   An **indicator random variable** is a discrete random variable defined in the following way (informally): $\mathbb{1}_A = 1$ if event A occurs, 0 otherwise. Show that:

   (a) $\mathbb{E}(\mathbb{1}_A) = \mathbb{P}(A)$.

   (b) If X is a non-negative random variable, then for $c > 0$, $\mathbb{P}(X \geq c) \leq \mathbb{E}(X)/c$. (Remark: This is known as Markov's Inequality.)
   [*Hint*: Consider the random variable $\mathbb{1}_{\{X \geq c\}}$.]

(c) Now suppose Y is a random variable (not necessarily non-negative). Provide an upper bound for $\mathbb{P}(Y \geq c)$ for $c > 0$.

5. **Generating Random Variables**

Consider a continuous random variable $U \sim \text{Uniform}[0, 1]$. Let $F : \mathbb{R} \to [0, 1]$ be a strictly increasing distribution function. Show that $F^{-1}(U)$ has the cumulative distribution function (CDF) F.

6. **Auction Theory**

This problem explores auction theory and is meant to be done at the same time as the lab.

In auction theory, n bidders have **valuations** which represent how much they value an item; we will make the simplifying assumption that the valuations are i.i.d. with density $f(x)$. In the first-price auction, the bidder who makes the highest bid wins the item and pays his/her bid. In the second-price auction, the bidder who makes the highest bid wins the auction, and pays an amount equal to the *second-highest* bid. A strategy for the auction is a **bidding function** $\beta(x)$, where x is the bidder's valuation. The bidding function determines how much to bid as a function of the bidder's valuation, and the goal is to find a bidding function $\beta(\cdot)$ which maximizes your expected utility (0 if you do not win, and your valuation minus the amount of money you bid if you do win).

(a) For the first-price auction, consider the following scenario: each person draws his/her valuation uniformly from the interval $(0, 1)$ (so $f(x) = 1$ for $x \in (0, 1)$). Suppose that the other bidders bid their own valuations (they use $\beta(x) = x$, the identity bidding function). Consider the case where there is only one other bidder. Your Stanford friend insists that you should always bid $\beta(x) = 1$. Your Berkeley friend tells your Stanford friend that it would be better to bid

$$\beta(x) = \frac{x}{2}.$$

Who is correct?

(b) Consider the same situation as the previous part, but now assume that there are n other bidders. Your Stanford friend again suggests that $\beta(x) = 1$ is the best bid. Your Berkeley friend suggests

$$\beta(x) = \frac{n}{n+1}x.$$

Who is correct this time?

(c) Consider a second-price auction where the bidders' valuations are i.i.d. with the exponential density (with parameter λ). Again, they use the identity bidding function, $\beta(x) = x$. What is the distribution of the price P at which the item sells?