UC Berkeley
Department of Electrical Engineering and Computer Sciences

Electrical Engineering 126: Probability and Random Processes

Problem Set 8
Fall 2017

Issued: October 28, 2017 **Due:** 9 AM, Thursday, November 2, 2017

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1. **Estimating an Exponential Distribution**

   You draw a sample $X_1, \dots, X_n$ (n is a positive integer) for the lifetime of a light bulb (assumed to be exponentially distributed).

   (a) Frankly, you are beginning to think that your boss is unreasonable. She makes the following demands:

   - You have exactly 3 samples, with $X_1 = 3$, $X_2 = 7$, $X_3 = 5$. That is, the first light bulb dies after 3 seconds, the second light bulb dies after 7 seconds, and the third light bulb dies after 5 seconds.
   - Your confidence interval is specified to be the interval $(0, \varepsilon)$.
   - You are no longer estimating the mean lifetime; now you are estimating the rate λ at which the light bulbs die.
   - Your confidence interval must be *exact*: you may not use any inequalities or approximations.

   What is your reported confidence, in terms of ε? [Please do not leave your answers in terms of an integral.]

   (b) In the same setting as the previous part, can you give an exact confidence interval for the mean light bulb lifetime with the same confidence level as before? (*Hint*: You don't need to complete Part (c) to answer this successfully.)

2. **Chernoff Bound Application: Load Balancing**

   Here, we will give an application for the Chernoff bound which is instrumental for calculating confidence intervals. However, we will need a slightly more general version of the bound that works for any Bernoulli random variables. For any positive integer n, if $X_1, \dots, X_n$ are i.i.d. Bernoulli, with $\mathbb{P}(X_i = 1) = p$, and $S_n = \sum_{i=1}^n X_i$, then the following bound holds for $0 \le \varepsilon \le 1$:

   $$\mathbb{P}\big(S_n > (1+\varepsilon)np\big) \le \exp\Big(-\frac{\varepsilon^2 np}{3}\Big). \tag{1}$$

   You may take (1) as a fact (or try to prove it on your own if you want!).

   Here is the setting: there are k (k a positive integer) servers and n users. The simplest load balancing scheme is simply to assign each user to a server chosen uniformly at random (think of the users as "balls" and we are tossing them into server "bins"). By using the union bound, show that with probability at least $1 - 1/k^2$, the maximum load of any server is at most $n/k + 3\sqrt{\ln k}\sqrt{n/k}$.

3. **Basic Properties of Jointly Gaussian Random Variables**

   Prove that a collection of jointly Gaussian random variables $X_1, \ldots, X_n$ (n is a positive integer) are independent if and only if they are uncorrelated. Also show that any linear combination of these random variables will be a Gaussian random variable. [*Hint*: For first part, use the characteristic function definition, and look at the covariance matrix for uncorrelated RVs.]

4. **Gaussian Hypothesis Testing**

   Consider a hypothesis testing problem that if $X = 0$, you observe a sample of $\mathcal{N}(\mu_0, \sigma^2)$, and if $X = 1$, you observe a sample of $\mathcal{N}(\mu_1, \sigma^2)$, where $\mu_0, \mu_1 \in \mathbb{R}$, $\sigma^2 > 0$. Find the Neyman-Pearson test for false alarm $\alpha \in (0, 1)$, that is, $\mathbb{P}(\hat{X} = 1 \mid X = 0) \leq \alpha$.

5. **Hypothesis Test for Uniform Distribution**

   If $X = 0$, $Y \sim \text{Uniform}[-1, 1]$ and if $X = 1$, $Y \sim \text{Uniform}[0, 2]$. Solve a hypothesis testing problem so that the probability of false alarm is less than or equal $\beta \in (0, 1)$.

6. **BSC Hypothesis Testing**

   You are testing a digital link that corresponds to a BSC with some error probability $\epsilon \in [0, 0.5)$. You observe n inputs and outputs of the BSC, where n is a positive integer. You want to solve a hypothesis problem to detect that $\epsilon > 0.1$ with a probability of false alarm at most equal to 0.05. Assume that n is very large and use the CLT.