

# Electrical Engineering 126: Probability & Random Processes

## Final Cheat Sheet

Fall 2018

## 1 Distributions

- $X \sim \text{Bernoulli}(p)$ ,  $p \in [0, 1]$ .

PMF:  $p_X(x) = p^x(1-p)^{1-x}$ ,  $x \in \{0, 1\}$ .

MGF:  $M_X(s) = 1 - p + p \exp s$ .

Moments:  $\mathbb{E}[X] = p$ ,  $\text{var } X = p(1-p)$ .

- $X \sim \text{Binomial}(n, p)$ ,  $n \in \mathbb{Z}_+$ ,  $p \in [0, 1]$ .

PMF:  $p_X(x) = \binom{n}{x} p^x(1-p)^{n-x}$ ,  $x \in \{0, \dots, n\}$ .

MGF:  $M_X(s) = (1-p + p \exp s)^n$ .

Moments:  $\mathbb{E}[X] = np$ ,  $\text{var } X = np(1-p)$ .

- $X \sim \text{Geometric}(p)$ ,  $p \in (0, 1)$ .

PMF:  $p_X(x) = pq^{x-1}$ ,  $x \in \mathbb{Z}_+$ ,  $q = 1-p$ .

MGF:  $M_X(s) = (p \exp s)/(1 - q \exp s)$ ,  $s < \ln(1/q)$ .

Moments:  $\mathbb{E}[X] = p^{-1}$ ,  $\text{var } X = q/p^2$ .

- $X \sim \text{Poisson}(\lambda)$ ,  $\lambda > 0$ .

PMF:  $p_X(x) = \lambda^x \exp(-\lambda)/x!$ ,  $x \in \mathbb{N}$ .

MGF:  $M_X(s) = \exp(\lambda(\exp s - 1))$ .

Moments:  $\mathbb{E}[X] = \lambda$ ,  $\text{var } X = \lambda$ .

$X, Y$  independent,  $X \sim \text{Poisson}(\lambda)$ ,  $Y \sim \text{Poisson}(\mu) \implies X + Y \sim \text{Poisson}(\lambda + \mu)$ .

- $X \sim \text{Uniform}[a, b]$ ,  $a < b$ .

PDF:  $f_X(x) = (b-a)^{-1}$ ,  $x \in [a, b]$ .

MGF:  $M_X(s) = (\exp(sb) - \exp(sa))/(s(b-a))$ .

Moments:  $\mathbb{E}[X] = (a+b)/2$ ,  $\text{var } X = (b-a)^2/12$ .

- $X \sim \text{Exponential}(\lambda)$ ,  $\lambda > 0$ .

PDF:  $f_X(x) = \lambda \exp(-\lambda x)$ ,  $x > 0$ .

CDF:  $F_X(x) = (1 - \exp(-\lambda x)) \mathbb{1}_{\{x \geq 0\}}$ .

MGF:  $M_X(s) = \lambda/(\lambda - s)$ ,  $s < \lambda$ .

Moments:  $\mathbb{E}[X] = \lambda^{-1}$ ,  $\text{var } X = \lambda^{-2}$ .

- $X \sim \mathcal{N}(\mu, \sigma^2)$ ,  $\mu \in \mathbb{R}$ ,  $\sigma^2 > 0$ .

PDF:  $f_X(x) = (\sqrt{2\pi}\sigma)^{-1} \exp\{-(x-\mu)^2/(2\sigma^2)\}$ .

CDF:  $F_X(x) = \Phi(x)$ .

MGF:  $M_X(s) = \exp(\mu s + \sigma^2 s^2/2)$ .

Moments:  $\mathbb{E}[X] = \mu$ ,  $\text{var } X = \sigma^2$ .

Gaussian Q-Function:  $Q(x) = P(X > x\sigma + \mu) \implies F_X(x) = Q\left(\frac{x-\mu}{\sigma}\right)$

$X, Y$  independent,  $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$ ,  $Y \sim \mathcal{N}(\mu_2, \sigma_2^2) \implies X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ .

Continued:

- $X \sim \text{Pascal}(k, p)$ ,  $k \in \mathbb{Z}_+$ ,  $p \in (0, 1)$ .

Sum of  $k$  i.i.d. Geometric( $p$ ).

PMF:  $p_X(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$ ,  $x = k, k+1, k+2, \dots$

- $X \sim \text{Erlang}(k, \lambda)$ ,  $k \in \mathbb{Z}_+$ ,  $\lambda > 0$ .

Sum of  $k$  i.i.d. Exponential( $\lambda$ ).

PDF:  $f_X(x) = \lambda^k x^{k-1} \exp(-\lambda x)/(k-1)!$ ,  $x \geq 0$ .

- $X \sim \mathcal{N}_n(\mu, \Sigma)$ ,  $n \in \mathbb{Z}_+$  (joint Gaussian).

PDF (assuming  $\Sigma$  invertible): For  $x \in \mathbb{R}^n$ ,  
 $f_X(x) = [(2\pi)^n \det \Sigma]^{-1/2} \exp\{-(x-\mu)^\top \Sigma^{-1} (x-\mu)/2\}$ .

MGF:  $M_X(s) = \mathbb{E}[\exp(s^\top X)] = \exp(\mu^\top s + s^\top \Sigma s/2)$ .

Moments:  $\mathbb{E}[X] = \mu$ ,  $\text{cov } X = \Sigma$ .

## 2 Definitions & Equations

Tail Sum: For  $X \geq 0$ ,  $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geq x) dx$ .

Variance:  $\text{var } X = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ . Sum:  
 $\text{var } \sum_{i=1}^n X_i = \sum_{i=1}^n \text{var } X_i + \sum_{i \neq j} \text{cov}(X_i, X_j)$ .

Covariance:  $\text{cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$ . Matrix: If  $X = (X_1, \dots, X_n)$ ,  $(\text{cov } X)_{i,j} = \text{cov}(X_i, X_j)$ .

Correlation:  $\rho(X, Y) = \text{cov}(X, Y)/\sqrt{(\text{var } X)(\text{var } Y)}$ .

Entropy:  $H(X) = -\sum_{x \in \mathcal{X}} p(x) \log_2 p(x) = -\mathbb{E}[\log_2 p(X)]$ .

Order Statistics:  $f_{X^{(i)}}(x) = n \binom{n-1}{i-1} f(x) F(x)^{i-1} (1 - F(x))^{n-i}$ .

MGF:  $M_X(s) = \mathbb{E}[\exp(sX)]$ .

Markov: For  $X \geq 0$ ,  $x > 0$ ,  $\mathbb{P}(X \geq x) \leq \mathbb{E}[X]/x$ .

Chebyshev: For  $x > 0$ ,  $\mathbb{P}(|X - \mathbb{E}[X]| \geq x) \leq (\text{var } X)/x^2$ .

Chernoff: For  $t > 0$ ,  $\mathbb{P}(X \geq x) = \mathbb{P}(e^{tX} \geq e^{tx})$ .  
For  $t > 0$ ,  $\mathbb{P}(X \leq x) = \mathbb{P}(e^{-tX} \geq e^{-tx})$ .

LLSE:  $L[X | Y] - \mathbb{E}[X] = [\text{cov}(X, Y)/(\text{var } Y)](Y - \mathbb{E}[Y])$ .

Kalman Dynamics: For  $n \in \mathbb{N}$ ,

$$\begin{aligned} X_{n+1} &= AX_n + V_n, \\ Y_n &= CX_n + W_n, \end{aligned}$$

where  $X_0, (V_n, n \in \mathbb{N}), (W_n, n \in \mathbb{N})$  are zero mean, orthogonal,  
with  $\text{cov } V_n = \Sigma_V$ ,  $\text{cov } W_n = \Sigma_W$  for all  $n \in \mathbb{N}$ .

Kalman Filter: Assume  $\Sigma_W$  is invertible.

$$\begin{aligned} \hat{X}_{n|n} &:= L[X_n | Y_0, Y_1, \dots, Y_n] = \hat{X}_{n|n-1} + K_n \tilde{Y}_n, \\ \hat{X}_{n|n-1} &:= L[X_n | Y_0, Y_1, \dots, Y_{n-1}] = A\hat{X}_{n-1|n-1}, \\ \tilde{Y}_n &:= Y_n - L[Y_n | Y_0, Y_1, \dots, Y_{n-1}] = Y_n - C\hat{X}_{n|n-1}, \\ K_n &= \Sigma_{n|n-1} C^\top (C\Sigma_{n|n-1} C^\top + \Sigma_W)^{-1}, \\ \Sigma_{n|n-1} &:= \text{cov}(X_n - \hat{X}_{n|n-1}) = A\Sigma_{n-1|n-1} A^\top + \Sigma_V, \\ \Sigma_{n|n} &:= \text{cov}(X_n - \hat{X}_{n|n}) = (I - K_n C)\Sigma_{n|n-1}. \end{aligned}$$