UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Discussion Session 14
Fall 2018

Problem 1. **Forrest Gump**
Forrest Gump is running across the United States, and we would like to track his progress. Assume that on day $n \in \mathbb{N}$ he runs $X(n)$ miles, and the amount he runs each day is determined by the amount he ran on the previous day with some random noise in the following manner: $X(n) = \alpha X(n-1) + V(n)$. Unfortunately, the measurements of the distance he traveled on each day are also subject to some noise. Assume that $Y(n)$ gives the measured number of miles Forrest Gump traveled on day n and that $Y(n) = \beta X(n) + W(n)$. For this problem, assume that $X(0) \sim \mathcal{N}(0, \sigma_X^2), W(n) \sim \mathcal{N}(0, \sigma_W^2), V(n) \sim \mathcal{N}(0, \sigma_V^2)$ are independent.

1. Suppose that you observe $Y(0)$. Find the MMSE of $X(0)$ given this observation.
2. Express both $\mathbb{E}[Y(n) \mid Y(0), \dots, Y(n-1)]$ and $\mathbb{E}[X(n) \mid Y(0), \dots, Y(n-1)]$ in terms of $\hat{X}(n-1)$, where $\hat{X}(n-1)$ is the MMSE of $X(n-1)$ given the observations $Y(0), Y(1), \dots, Y(n-1)$.
3. Show that:
$$\hat{X}(n) = \alpha\hat{X}(n-1) + k_n[Y(n) - \alpha\beta\hat{X}(n-1)]$$
where
$$k_n = \frac{\operatorname{cov}(X(n), \tilde{Y}(n))}{\operatorname{var}\tilde{Y}(n)}$$
and $\tilde{Y}(n) = Y(n) - L[Y(n) \mid Y(0), Y(1), \dots, Y(n-1)]$.

Problem 2. **Hidden Markov Models**
A hidden Markov model (HMM) is a Markov chain $\{X_n\}_{n=0}^{\infty}$ in which the states are considered "hidden" or "latent". In other words, we do not directly observe $\{X_n\}_{n=0}^{\infty}$. Instead, we observe $\{Y_n\}_{n=0}^{\infty}$, where $Q(x, y)$ is the probability that state x will emit observation y. π_0 is the initial distribution for the Markov chain, and P is the transition matrix.

1. What is $\Pr(X_0 = x_0, Y_0 = y_0, \dots, X_n = x_n, Y_n = y_n)$, where n is a positive integer, $x_0, \dots, x_n$ are hidden states, and $y_0, \dots, y_n$ are observations?
2. What is $\Pr(X_0 = x_0 \mid Y_0 = y_0)$?
3. We observe $(y_0, \dots, y_n)$ and we would like to find the most likely sequence of hidden states $(x_0, \dots, x_n)$ which gave rise to the observations. Let
$$U(x_m, m) = \max_{x_{m+1},\dots,x_n \in \mathcal{X}} \Pr(X_m = x_m, X_{m+1:n} = x_{m+1:n}, Y_{0:n} = y_{0:n})$$

denote the largest probability for a sequence of hidden states beginning at state x_m at time $m \in \mathbb{N}$, along with the observations $(y_0, \ldots, y_n)$. Develop a recursion for $U(x_m, m)$ in terms of $U(x_{m+1}, m+1)$, $x_{m+1} \in \mathcal{X}$.