UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Discussion Session 4
Fall 2018

Problem 1. **Exponential Bounds** Let $X \sim \text{Exp}(\lambda)$. For $x > \lambda^{-1}$, calculate bounds on $\Pr(X \geq x)$ using Markov's Inequality and the Chebyshev's Inequality.

Problem 2. **First Time to Decrease**
Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent and identically distributed (i.i.d.) continuous random variables with common PDF f.

1. Argue that $\Pr(X_i = X_j) = 0$ for $i \neq j$.
2. Calculate $\Pr(X_1 \leq X_2 \leq \cdots \leq X_{n-1})$.
3. Let N be a random variable which is equal to the first time that the sequence of the random variables will decrease, i.e.
$$N = \min\{n \in \mathbb{Z}_{\geq 2} \mid X_{n-1} > X_n\}.$$
Calculate $\mathbb{E}[N]$.

Problem 3. **Second Moment Method**
Consider a non-negative RV Y, with $\mathbb{E}(Y^2) < \infty$. Show that
$$\mathbb{P}(Y > 0) \geq \frac{\mathbb{E}(Y)^2}{\mathbb{E}(Y^2)}.$$
Hint: Use Cauchy-Schwarz on $Y 1_{\{Y>0\}}$.

Problem 4. **[Bonus] Gaussian Tail Bounds**
Let $\phi(y) = \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$ be the PDF of a standard normal random variable $Y \sim \mathcal{N}(0,1)$.

1. Show that for $y \neq 0$ we have that
$$\phi(y) = -\frac{1}{y} \cdot \phi'(y).$$
2. Use (a) to show that
$$\Pr(Y \geq t) \leq \frac{1}{t} \cdot \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}, \quad \text{for all } t > 0.$$
3. Use part (a) to show that
$$\left(\frac{1}{t} - \frac{1}{t^3}\right) \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} \leq \Pr(Y \geq t), \quad \text{for all } t > 0.$$