UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Discussion Session 8
Fall 2018

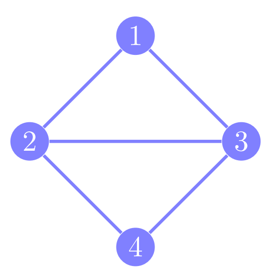
Problem 1. **Capacity of Symmetric Channels**
Consider a communication channel with input X and output Y, both on the alphabet $\{1, 2, 3\}$, with $P_{Y|X}$ given as

$$P_{Y|X}(y|x) = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.3 & 0.2 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}$$

Here, rows correspond to X and columns correspond to Y. Therefore, for instance, $P_{Y|X}(2|1) = 0.2$ and $P_{Y|X}(1|2) = 0.5$. Such a channel is called "symmetric", because all rows of the probability transition matrix are permutations of each other, and so are the columns.

1. Show that irrespective of the input distribution P_X, $H(Y|X)$ is always constant. Call this constant c and find it.
2. Show that $I(X; Y) \leq \log 3 - c$, with the constant c found in the previous part.
3. Find a input distribution $P_X(x)$ for which $I(X; Y) = \log 3 - c$.

Problem 2. **Conditional entropy of a random walk**
Consider a Markov chain which is random walk on the following graph



Here at time k, if we are at node $1 \leq v \leq 4$, we chose X_{k+1} to be one of the neighbors of v uniformly at random.

1. Show that the chain has a unique stationary distribution π and find it.
2. Assume that we initialize the chain at the stationary distribution, i.e. $X_0 \sim \pi$. Find $I(X_0; X_1)$.

Problem 3. **Generating Erdös-Renyi Random Graphs**
True/False: Let G_1 and G_2 be independent Erdös-Renyi random graphs on n vertices with probabilities p_1 and p_2, respectively. Let $G = G_1 \cup G_2$, that is, G is generated by combining the edges from G_1 and G_2. Then, G is an Erdös-Renyi random graph on n vertices with probability $p_1 + p_2$.

Problem 4. **Random Graph**
Consider a random undirected graph on n vertices, where each of the $\binom{n}{2}$ possible edges is present with probality p independently of all the other edges. If $p = 0$ we have a fully empty graph with n completely disconnected vertices; in contrast, if $p = 1$, every edge exists, the network is an n-clique, and every vertex is a distance one from every other vertex.

1. Fix a particular vertex of the graph, and let D be a random variable which is equal to the degree of this vertex. What is the PMF of D? Calculate $\lambda \triangleq \mathbb{E}[D]$.
2. Assume that $p = c/n$ where $c > 0$ is a constant, independent of n. For large values of n, how you would approximate the PMF of D?