UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Discussion Section 9
Fall 2018

Problem 1. **Poisson Practice**
Let $(N(t), t \geq 0)$ be a Poisson process with rate λ. Let T_k denote the time of k-th arrival, for $k \in \mathbb{N}$, and given $0 \leq s < t$, we write $N(s,t) = N(t) - N(s)$. Compute:

1. $\mathbb{P}(N(1) + N(2,4) + N(3,5) = 0)$.
2. $\mathbb{E}(N(1,3) \mid N(1,2) = 3)$.
3. $\mathbb{E}(T_2 \mid N(2) = 1)$.

Solution 1.

1. The event $\{N(1) + N(2,4) + N(3,5) = 0\}$ is the same as the intersection of 2 events, $\{N(1) = 0\}$ and $\{N(2,5) = 0\}$. These are independent with probabilities $\exp(-\lambda)$ and $\exp(-3\lambda)$. Hence

$$\mathbb{P}[N(1) + N(2,4) + N(3,5) = 0] = \exp(-4\lambda).$$

2. $N(1,3) = N(1,2) + N(2,3)$. We know $N(2,3)$ is independent of $N(1,2)$. Hence, $\mathbb{E}(N(1,3) \mid N(1,2) = 3) = 3 + \lambda$.

3. Since $N(2) = 1$, the second interarrival time T_2 hasn't lapsed yet at $t = 2$. From the memoryless property of the exponential distribution:

$$\mathbb{E}\big(T_2 - 2 \bigm| N(2) = 1\big) = \frac{1}{\lambda}.$$

Hence the answer is $2 + \lambda^{-1}$.

Problem 2. **Customers in a Store**
Consider two independent Poisson processes with rates λ_1 and λ_2. Those processes measure the number of customers arriving in store 1 and 2.

1. What is the probability that a customer arrives in store 1 before any arrives in store 2?
2. What is the probability that in the first hour exactly 6 customers arrive, in total, at the two stores?
3. Given that exactly 6 have arrived, in total, at the two stores, what is the probability that exactly 4 went to store 1?

Solution 2. 1. **Solution 1:** Consider the sum of two processes which is a Poisson process with rate $\lambda_1 + \lambda_2$. You mark each customer in this process as 1 with probability $\lambda_1/(\lambda_1+\lambda_2)$ and mark as 2 otherwise. The resulting two processes are Poisson processes of rates λ_1 and λ_2. Thus, the probability of having the first customer going to store 1 is equal to the probability of marking the first customer as 1 which is

$$\frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Solution 2: The arrival times of the first customer of the two stores are $X \sim \mathit{Exponential}(\lambda_1)$ and $Y \sim \mathit{Exponential}(\lambda_2)$, respectively. Then using the total probability theorem we have that

$$\begin{aligned}
\Pr(X < Y) &= \int_0^\infty f_Y(y) \Pr(X < Y \mid Y = y)\, dy \\
&= \int_0^\infty \lambda_2 \exp^{-\lambda_2 y}(1 - \exp^{-\lambda_1 y})\, dy \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\end{aligned}$$

2.

$$\frac{\exp^{-(\lambda_1+\lambda_2)}(\lambda_1 + \lambda_2)^6}{6!}.$$

3.

$$\binom{6}{4}\Big(\frac{\lambda_1}{\lambda_1 + \lambda_2}\Big)^4\Big(\frac{\lambda_2}{\lambda_1 + \lambda_2}\Big)^2.$$

Problem 3. **Minimum and Maximum of Exponentials**
Let $\lambda_1, \lambda_2 > 0$, and $X_1 \sim \mathit{Exponential}(\lambda_1)$, $X_2 \sim \mathit{Exponential}(\lambda_2)$ are independent. Also, define $U := \min(X_1, X_2)$ and $V := \max(X_1, X_2)$. Show that U and $V - U$ are independent.

Solution 3. For $u, w > 0$,

$$\begin{aligned}
&\Pr(U \le u, V - U \le w, X_1 < X_2) = \Pr(X_1 \le u, X_1 < X_2 \le X_1 + w) \\
&= \int_0^u \int_{x_1}^{x_1+w} \lambda_2 \exp(-\lambda_2 x_2)\, dx_2\, \lambda_1 \exp(-\lambda_1 x_1)\, dx_1 \\
&= \int_0^u \big\{\exp(-\lambda_2 x_1) - \exp\big(-\lambda_2(x_1 + w)\big)\big\}\lambda_1 \exp(-\lambda_1 x_1)\, dx_1 \\
&= \big(1 - \exp(-\lambda_2 w)\big) \int_0^u \lambda_1 \exp\big(-(\lambda_1 + \lambda_2)x_1\big)\, dx_1 \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}\big(1 - \exp\{-(\lambda_1 + \lambda_2)u\}\big)\big(1 - \exp(-\lambda_2 w)\big).
\end{aligned}$$

By symmetry, interchanging the roles of 1 and 2 yields

$$\begin{aligned}
&\Pr(U \le u, V - U \le w, X_2 < X_1) \\
&\quad = \frac{\lambda_2}{\lambda_1 + \lambda_2}\big(1 - \exp\{-(\lambda_1 + \lambda_2)u\}\big)\big(1 - \exp(-\lambda_1 w)\big).
\end{aligned}$$

Adding these two expressions yields

$$\Pr(U \le u, V - U \le w) = \big(1 - \exp\{-(\lambda_1 + \lambda_2)u\}\big)p_w, \qquad \text{where}$$
$$p_w := \frac{\lambda_1}{\lambda_1 + \lambda_2}\big(1 - \exp(-\lambda_2 w)\big) + \frac{\lambda_2}{\lambda_1 + \lambda_2}\big(1 - \exp(-\lambda_1 w)\big).$$

The joint CDF splits into a product of factors $\Pr(U \le u)\Pr(V - U \le w)$ which proves independence. To interpret the second term, observe that $\lambda_1/(\lambda_1 + \lambda_2)$ is the probability of the event $\{X_1 < X_2\}$; and conditioned on this event, $V - U \sim$ *Exponential*(λ_2) by the memoryless property.

Problem 4. **Bonus: Random Telegraph Wave**
Let $\{N_t,\ t \ge 0\}$ be a Poisson process with rate λ and define $X_t = X_0(-1)^{N_t}$ where $X_0 \in \{0, 1\}$ is a random variable independent of N_t.

(a) Does the process X_t have independent increments?

(b) Calculate $\Pr(X_t = 1)$ if $\Pr(X_0 = 1) = p$.

(c) Assume that $p = 0.5$. Calculate $\mathbb{E}[X_{t+s}X_s]$ for $s, t \ge 0$.

Solution 4. (a) No, the process does not have independent increments. According to the definition of independent increments, for any $0 < t_0 < t_1 < t_2$, we should have $X_{t_2} - X_{t_1}$ is independent of $X_{t_1} - X_{t_0}$. However, suppose $X_0 = 1$ and $X_{t_1} - X_{t_0} = 2$. This means that from t_0 to t_1, X_t increases from -1 to 1. Then it is impossible to have $X_{t_2} - X_{t_1} = 2$ since $X_t \in \{-1, 1\}$ for all $t > 0$, when $X_0 = 1$.

(b) First we calculate $\Pr(N_t \text{ is even})$.

$$\Pr(N_t \text{ is even}) = \sum_{i=0,\ i \text{ is even}}^{\infty} \frac{(\lambda t)^i \exp^{-\lambda t}}{i!} = \frac{\exp^{-\lambda t}}{2}\Big(\sum_{i=0}^{\infty}\frac{(\lambda t)^i}{i!} + \sum_{i=0}^{\infty}\frac{(-\lambda t)^i}{i!}\Big)$$
$$= \frac{\exp^{-\lambda t}}{2}(\exp^{\lambda t} + \exp^{-\lambda t}) = \frac{1 + \exp^{-2\lambda t}}{2}.$$

$$\Pr(X_t = 1) = p\Pr(N_t \text{ is even}) = p\frac{1 + \exp^{-2\lambda t}}{2}.$$

(c) If $X_0 = 0$, obviously there is $\mathbb{E}[X_{t+s}X_s] = 0$ for all $s, t \ge 0$. For $X_0 = 1$, we have

$$\Pr(X_{t+s}X_s = 1) = \Pr(N_{t+s} - N_s \text{ is even}) = \Pr(N_t \text{ is even})$$
$$= \frac{1}{2}(1 + \exp^{-2\lambda t}),$$

and

$$\Pr(X_{t+s}X_s = -1) = \Pr(N_{t+s} - N_s \text{ is odd}) = \Pr(N_t \text{ is odd})$$
$$= \frac{1}{2}(1 - \exp^{-2\lambda t}).$$

Therefore, we get

$$\begin{aligned}\mathbb{E}[X_{t+s}X_s] &= \frac{1}{2}\mathbb{E}[X_{t+s}X_s \mid X_0 = 1] \\ &= \frac{1}{2}\Big[\frac{1}{2}(1+\exp^{-2\lambda t}) - \frac{1}{2}(1-\exp^{-2\lambda t})\Big] = \frac{1}{2}\exp^{-2\lambda t}.\end{aligned}$$