UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: PROBABILITY AND RANDOM PROCESSES

Problem Set 1

Fall 2018

Issued: Thursday, August 23, 2018 **Due:** Wednesday, August 29 7, 2018

Problem 1. (i) Show that the probability that exactly one of the events A and B occurs is $\mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B)$.

(ii) If A is independent of itself, show that $\mathbb{P}(A) = 0$ or 1.

Problem 2.

Alice and Bob have $2n + 1$ fair coins, each coin with probability of heads equal to 1/2. Bob tosses $n + 1$ coins, while Alice tosses the remaining n coins. Assuming independent coin tosses , show that the probability that after all coins have been tossed , Bob will have gotten more heads than Alice is 1/2 (Hint: use symmetry).

Problem 3. Each of k jars contains w white and b black balls. A ball is randomly chosen from jar 1 and transferred to jar 2, then a ball is randomly chosen from jar 2 and transferred to jar 3, etc. Finally, a ball is randomly chosen from jar k. Show that the probability that the last ball is white is the same as the probability that the first ball is white, i.e., it is $w/(w + b)$.

Problem 4. There are N passengers in a plane with N assigned seats, but after boarding, the passengers take the seats randomly. Assuming all seating arrangements are equally likely, what is the probability that no passenger is in their assigned seat? Compute the probability when $N \to \infty$.
Hint: Use the inclusion-exclusion principle.

Problem 5. (i) Let $n \in \mathbb{Z}_{>0}$ and $A_1, \ldots, A_n$ be any events. Prove the **union bound**: $\Pr(\bigcup_{i=1}^n A_i) \leq \sum_{i=1}^n \Pr(A_i)$ (Hint: Use Induction).

(ii) Let $A_1 \subseteq A_2 \subseteq \cdots$ be a sequence of increasing events. Prove that $\lim_{n\to\infty} \Pr(A_n) = \Pr(\bigcup_{i=1}^\infty A_i)$. [This can be viewed as a **continuity** property for Probability.]

(iii) Let $A_1, A_2, \ldots$ be a sequence of events. Prove that the union bound holds for the following setting: $\Pr(\bigcup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \Pr(A_i)$.

Problem 6. (i) Superman and Captain America are playing a game of basketball. At the end of the game, Captain America scored n points and Superman scored m points, where $n > m$ are positive integers. Supposing that each basket counts for exactly one point, what is the probability that after the start of the game (when they are initially tied), Captain America was always *strictly* ahead of Superman? (Assume that all sequences of baskets which result in the final score of n baskets for Captain America and m baskets for Superman are equally likely.)(Hint: Think about symmetry. First, try to figure out which is more likely: there was a tie and Superman scored the first point, or there was a tie and Captain America scored the first point?)

Problem 7 (**Bonus**). In a *tournament* with n players (where n is a positive integer), each player plays against every other player for a total of $\binom{n}{2}$ games (assume that there are no ties). Let k be a positive integer. Is it always possible to find a tournament such that for any subset A of k players, there is a player who has beaten everyone in A? For such a tournament, let us say that every k-subset is dominated. For example, Figure 1 depicts the smallest tournament in which every 2-subset is dominated.

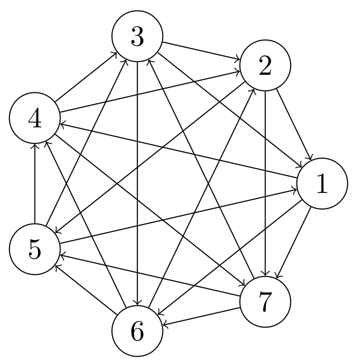

Figure 1: A tournament with 7 vertices such that every pair of players is beaten by a third player.

In fact, as long as $\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$, it is possible to find a tournament of n players such that every k-subset is dominated. Prove this fact, and explain why it implies that for any positive integer k there exists a tournament such that every k-subset is dominated.

Note: The bonus question is just for fun. You are not required to submit the bonus question, but do give it a try and write down your progress.