UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 10
Fall 2018

Issued: Wednesday, October 24, 2018 **Due:** Wednesday, October 31, 2018

Problem 1. **Basketball II**
Team A and Team B are playing an untimed basketball game in which the two teams score points according to independent Poisson processes. Team A scores points according to a Poisson process with rate λ_A and Team B scores points according to a Poisson process with rate λ_B. The game is over when one of the teams has scored k more points than the other team. Find the probability that Team A wins.

Problem 2. **Continuous-Time Markov Chains: Introduction**
Consider the continuous-time Markov process with state space $\{1, 2, 3, 4\}$ and the rate matrix

$$Q = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & -3 & 2 & 1 \\ 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

1. Find the stationary distribution p of the Markov process.
2. Find the stationary distribtion π of the jump chain, i.e., the discrete-time Markov chain which only keeps track of the jumps of the CTMC. Formally, if the CTMC $(X(t))_{t\geq 0}$ jumps at times $T_1, T_2, T_3, \ldots$, then the DTMC is defined as $(Y_n)_{n=1}^{\infty}$ where $Y_n := X_{T_n}$.
3. Suppose the chain starts in state 1. What is the expected amount of time until it changes state for the first time?
4. Again assume the chain starts in state 1. What is the expected amount of time until the chain is in state 4?

Problem 3. **Links**
Consider the graph shown in Figure 1. There are two parallel directed paths from the source node S to the destination node D. One of these is the path comprised of the two successive directed links 1 and 2, going through the intermediate node I. The other is the direct path comprised of the directed link 3. At any time, each of the links is in one of two state: *on* or *off*. Link i switches between its on and off states at rate λ_i, $i = 1, 2, 3$, independently over links. The rates here refer to the rate in exponential distribution. Thus the state of each link can be modeled as a

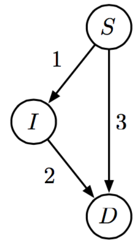

Figure 1: Source to destination links.

two-state continuous-time Markov chain. We will say that S is connected to D at any time iff there is a path from S to D comprised of on links.

(a) What is the stationary probability that S is connected to D?

(b) Assume that the process is in stationarity. Condition on S being connected to D at time 0, and let δ_t, for $t > 0$, denote the conditional probability that S is not connected to D at time t. What is $\left.\frac{\mathrm{d}}{\mathrm{d}t}\delta_t\right|_{t=0}$ (i.e., the first derivative of δ_t at time 0)?

(c) Assume that the process is in stationarity. Condition on S being not connected to D at time 0. What is the conditional mean time it takes for S to be connected to D?

Problem 4. $M/M/2$ **Queue**

A queue has Poisson arrivals with rate λ. It has two servers that work in parallel. When there are at least two customers in the queue, two are being served. When there is only one customer, only one server is active. The service times are i.i.d. exponential random variables with rate μ. Let $X(t)$ be the number of customers either in the queue or in service at time t.

1. Argue that the process $(X(t), t \geq 0)$ is a Markov process.
2. Draw the state transition diagram.
3. Find the range of values of μ for which the Markov chain is positive-recurrent and for this range of values calculate the stationary distribution of the Markov chain.

Problem 5. **Two-Server System**

A company has two servers (the second server is a backup in case the first server fails, so only one server is ever used at a time). When a server is running, the time until it breaks down is exponentially distributed with rate μ. When a server is broken, it is taken to the repair shop. The repair shop can only fix one server at a time, and its repair time is exponentially distributed with rate λ. Find the long-run probability that no servers are operational.

Problem 6. **Taxi Queue**

Empty taxis pass by a street corner at a Poisson rate of two per minute and pick up a passenger if one is waiting there. Passengers arrive at the street corner at a Poisson rate of one per minute and wait for a taxi only if there are less than four persons waiting; otherwise they leave and never return. John arrives at the street corner at a given time. Find his expected waiting time, given that he joins the queue. Assume that the process is in steady state.