UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 13

Fall 2018

Issued: Thursday, November 15, 2018 **Due:** Wednesday, November 21, 2018

Problem 1. **Geometric MMSE**
Let N be a geometric random variable with parameter $1-p$, and $(X_i)_{i\in\mathbb{N}}$ be i.i.d. exponential random variables with parameter λ. Let $T = X_1 + \cdots + X_N$. Compute the LLSE and MMSE of N given T.

Problem 2. **Property of MMSE**
Let $X, Y_1, \ldots, Y_n$ be square integrable random variables. Argue that

$$\mathbb{E}\big[(X - \mathbb{E}[X \mid Y_1, \ldots, Y_n])^2\big] \le \mathbb{E}\left[\left(X - \sum_{i=1}^{n} \mathbb{E}[X \mid Y_i]\right)^2\right].$$

Problem 3. **Gaussian Random Vector MMSE**
Let

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$$

be a Gaussian random vector.
Let

$$W = \begin{cases} 1, & \text{if } Y > 0 \\ 0, & \text{if } Y = 0 \\ -1, & \text{if } Y < 0 \end{cases}$$

be the sign of Y. Find $\mathbb{E}[WX \mid Y]$.

Problem 4. **Gaussian Estimation**
Let $Y = X + Z$ and $U = X - Z$, where X and Z are i.i.d. $\mathcal{N}(0,1)$.

(a) Find the joint distribution of U and Y.

(b) Find the MMSE of X given the observation Y, call this $\hat{X}(Y)$.

(c) Let the estimation error $E = X - \hat{X}(Y)$. Find the conditional distribution of E given Y.

Problem 5. **Stochastic Linear System MMSE**
Let $(V_n,\ n \in \mathbb{N})$ be i.i.d. $\mathcal{N}(0, \sigma^2)$ and independent of $X_0 = \mathcal{N}(0, u^2)$. Let $|a| < 1$. Define

$$X_{n+1} = aX_n + V_n, \qquad n \in \mathbb{N}.$$

1. What is the distribution of X_n, where n is a positive integer?
2. Find $\mathbb{E}[X_{n+m} \mid X_n]$ for $m, n \in \mathbb{N}$, $m \geq 1$.
3. Find u so that the distribution of X_n is the same for all $n \in \mathbb{N}$.

Problem 6. **Noisy Guessing**
Let X, Y, and Z be i.i.d. with the standard Gaussian distribution. Find $\mathbb{E}[X \mid X + Y, X + Z, Y - Z]$.
Hint: Argue that the observation $Y - Z$ is redundant.

Problem 7. **Bonus: Projections in Hilbert Space**
The following exercises are from the note on the Hilbert space of random variables. See the notes for some hints.

1. Let $\mathcal{H} := \{X : X \text{ is a real-valued random variable with } \mathbb{E}[X^2] < \infty\}$. Prove that $\langle X, Y\rangle := \mathbb{E}[XY]$ makes $\mathcal{H}$ into a real inner product space.
2. Let U be a subspace of a real inner product space V and let P be the projection map onto U. Prove that P is a linear transformation.
3. Suppose that U is finite-dimensional, $n := \dim U$, with basis $\{v_i\}_{i=1}^n$. Suppose that the basis is orthonormal. Show that $Py = \sum_{i=1}^n \langle y, v_i\rangle v_i$. (Note: If we take $U = \mathbb{R}^n$ with the standard inner product, then P can be represented as a matrix in the form $P = \sum_{i=1}^n v_i v_i^T$.)