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UC Berkeley

Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 14

Fall 2018

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Issued: Tuesday, November 27, 2018 **Due:** Tuesday, December 4, 2018

Problem 1. **Kalman Filter**

Assume that the following dynamics are given, for $n \in \mathbb{N}$,

$$X(n + 1) = AX(n) + V(n),$$
$$Y(n) = X(n) + W(n).$$

Y is the observation. V and W are independent Gaussian noise variables with mean zero and variance σ_V^2 and σ_W^2 respectively.

1. Write down the Kalman filter recursive equations for this system.

2. Let k be a positive integer. Compute the prediction $\mathbb{E}(X(n+k) \mid Y^{(n)})$, where $Y^{(n)}$ is the history of the observations $Y_0, \ldots, Y_n$, in terms of the estimate $\hat{X}(n) := \mathbb{E}(X(n) \mid Y^{(n)})$.

3. Now let $k = 1$ and compute the smoothing estimate $\mathbb{E}(X(n) \mid Y^{(n+1)})$ in terms of the quantities that appear in the Kalman filter equation.

   Hint: Use the law of total expectation

   $$\mathbb{E}\big(X(n) \mid Y^{(n+1)}\big) = \mathbb{E}\big[\mathbb{E}\big(X(n) \mid X(n+1), Y^{(n+1)}\big) \mid Y^{(n+1)}\big],$$

   as well as the *innovation*

   $$\tilde{X}(n+1) := X(n+1) - L[X(n+1) \mid Y^{(n)}].$$

Problem 2. **Hidden Markov Models**

Figure 1 shows the life of Sinho. Some days he is Tired and some days he is Energetic. But he doesn't tell you whether he's Tired or not, and all you can observe is whether he Jumps, Eats, Runs, or Sleeps. We start on day 1 in the Energetic state and there is one transition per day.

For the questions below, we use the following notations:

- q_t: state on day t

- O_t: observation on day t

1. What is $\Pr(q_2 = \text{Energetic} \mid O_2 = \text{Eat})$?

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2. What is $\Pr(O_3 = \text{Sleep} \mid O_2 = \text{Eat})$?

Problem 3. **Most Likely Sequence of States**
In this problem, we give an example of an HMM and a sequence of observations which demonstrates that the most likely sequence of hidden states (i.e., the output of the Viterbi algorithm) is *not* the same as computing the most likely state at each time. Your task is to verify that the following example works:
Consider a HMM with two states $\{0, 1\}$ and the hidden state is observed through a BSC with error probability $1/3$. The hidden state transitions according to $P(0, 0) = P(1, 1) = 3/4$. Assume that the initial state is equally likely to be 0 or 1. We see the observation 0 at time 0 and 1 at time 1.

Problem 4. **(Bonus) Error of the Kalman Filter for a Linear Stochastic System**
The linear stochastic system

$$\begin{bmatrix} X_{1,k+1} \\ X_{2,k+1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} X_{1,k} \\ X_{2,k} \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} w_k, \qquad k \geq 0,$$

starts from an arbitrary (known) initial condition $\begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix}$ and the system noise variables (w_k, $k \geq 0$) are i.i.d. normal with mean 0 and variance 1.
The state variables are not directly observable. However, we can observe

$$Y_k = X_{1,k} + X_{2,k}, \qquad k \geq 0.$$

Let $\hat{X}_{k|k}$ denote the minimum mean square error estimator of $X_k = \begin{bmatrix} X_{1,k} \\ X_{2,k} \end{bmatrix}$ given $(Y_0, \ldots, Y_k)$. Determine the asymptotic behavior of the covariance matrix of the estimation error.
Note: This problem needs thought. Note that there is no observation noise, so the assumption used in the derivation of the Kalman filter equations, that the covariance matrix of the observation noise is positive definite, is no longer valid.

Problem 5. **(Bonus) EM for Censored Exponential Data**
A common application of the EM algorithm is for **censored data** in statistics. Let n be a fixed positive integer denoting the sample size; let $X_1, \ldots, X_n$ be i.i.d. Exponential(λ) random variables; let $c_1, \ldots, c_n$ be known positive constants, and suppose that we observe $Y_i := \mathbb{1}\{X_i > c_i\}$ for each $i = 1, \ldots, n$. In other words, we do not get to observe the actual values of $X_1, \ldots, X_n$. We only get to observe whether the ith data point is greater than the level c_i. We would like to find the MLE for the rate λ. If we knew the values of $X_1, \ldots, X_n$, then we would use $\hat{\lambda} := n/(\sum_{i=1}^n X_i)$.
Applying the EM algorithm to the following problem, we will alternate between the following steps. First, initialize a guess $\hat{\lambda}^{(0)}$. Then, for $t = 0, 1, 2, \ldots$:

- **E step**: Compute $\bar{X}^{(t)} := \mathbb{E}_{\hat{\lambda}^{(t)}}[n^{-1}\sum_{i=1}^{n} X_i \mid Y_1, \ldots, Y_n]$, where the notation $\mathbb{E}_{\hat{\lambda}^{(t)}}$ means you should calculate the expectation as if

$$X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Exponential}(\hat{\lambda}^{(t)}).$$

- **M step**: The next estimate of the parameter is $\hat{\lambda}^{(t+1)} := 1/\bar{X}^{(t)}$.

(Do not worry about why the E and M steps look the way they do.)

1. Verify that the MLE estimate of λ given $X_1, \ldots, X_n$ is $\hat{\lambda} = n/(\sum_{i=1}^{n} X_i)$.

2. Explicitly write out what the E step looks like.

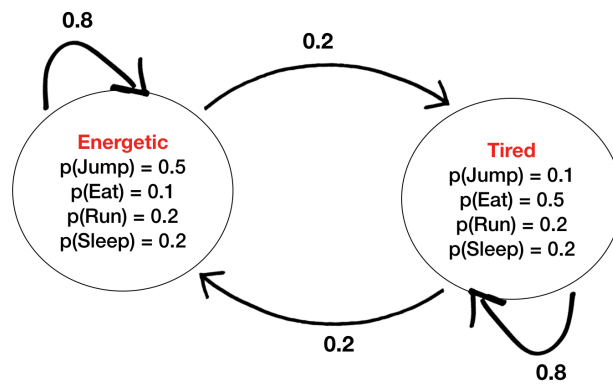3. Write out the joint PMF for the observations $Y_1, \ldots, Y_n$. Is it possible to find the MLE for λ given $Y_1, \ldots, Y_n$ directly?

Figure 1: HMM model for Sinho.