UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 3
Fall 2018

Issued: Thursday, September 6, 2018 **Due:** Wednesday, September 12, 2018

Problem 1. (**Conditional Distribution**) Let X have a Poisson distribution with parameter $\lambda > 0$. Suppose λ itself is random, having an exponential density with parameter θ.

(a) What is the distribution of X?

(b) Determine the conditional density of λ given $X = k$.

Problem 2. (**Record Score**) Let $X_1, X_2, \ldots, X_n$ be a sequence of iid continuous random variable with common probability density $f_X(x)$. For $n \geq 2$, define X_n as a record to date of the sequence X_i if $X_n > X_i$ for all $i < n$.

(a) Find the probability that X_2 is a record to date.
Hint: You should be able to do it without rigorous computation.

(b) Find the probability that X_n is record to date.

(c) Find the expected number of records to date that occur over the first m trials (Hint: Use indicator functions). Compute this when $m \to \infty$.

Problem 3. **Exponential Distribution with Floor** Let $W \sim exp(\lambda)$ for $\lambda > 0$, and let $X := \lfloor W \rfloor$ and $Y := W - X$.

(a) Find the PMF of X.

(b) For $y \in (0, 1)$ and $x \in \mathbb{N}$, find $Pr(Y \leq y \mid X = x)$. Find the CDF of Y.

(c) Find $\mathbb{E}[Y]$ and $Var(Y)$. (*Hint*: There is an easy way to do this problem, and a very tedious way to do this problem.)

(d) For a random vector $(X_1, \ldots, X_n)$ where n is a positive integer and $X_1, \ldots, X_n$ are real-valued random variables, the **covariance matrix** of $(X_1, \ldots, X_n)$ is the $n \times n$ matrix whose (i, j) entry is $cov(X_i, X_j)$ for all $i, j \in \{1, \ldots, n\}$. Find the covariance matrix of (W, Y).

Problem 4. (**Joint density**)

(a) If $X \sim Exp(\lambda)$ and $Y \sim Exp(\mu)$, X and Y independent, compute $\mathbb{P}(X < Y)$.

(b) If X_k, $1 \le k \le n$ are exponentially distributed with parameters $\lambda_1, \dots, \lambda_n$, show that,

$$\mathbb{P}(X_i = \min_{1 \le k \le n} X_k) = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}$$

Problem 5. **Gaussian Densities**

(a) Let $X_1 \sim \mathcal{N}(0, \sigma_1^2)$, $X_2 \sim \mathcal{N}(0, \sigma_2^2)$, where X_1 and X_2 are independent. Convolve the densities of X_1 and X_2 to show that $X_1 + X_2 \sim \mathcal{N}(0, \sigma_1^2 + \sigma_2^2)$.

(b) Show that all linear combinations of i.i.d. finitely many Gaussians are Gaussian.

(c) Let $X \sim \mathcal{N}(0, \sigma^2)$; find $\mathbb{E}[X^n]$ for $n \in \mathbb{N}$.

(d) Let $Z \sim \mathcal{N}(0, 1)$. Find the mean and covariance matrix of $(Z, \mathbf{1}\{Z > c\})$ in terms of ϕ and Φ, the standard Gaussian PDF and CDF respectively.

Problem 6. (Triangle density) Consider random variables X and Y which have a joint PDF uniform on the triangle with vertices at $(0,0), (1,0), (0,1)$.

(a) Find the joint PDF of X and Y.

(b) Find the marginal PDF of Y.

(c) Find the conditional PDF of X given Y.

(d) Find $E[X]$ in terms of $E[Y]$

(e) Find $E[X]$.

Problem 7. **Auction Theory**

This problem explores auction theory and we will have a related lab assignment.
In auction theory, n bidders (n is a positive integer) have **valuations** which represent how much they value an item; we will make the simplifying assumption that the valuations are i.i.d. with continuous density f. In the **first-price auction**, the bidder who makes the highest bid wins the item and pays his/her bid. In the **second-price auction**, the bidder who makes the highest bid wins the auction, and pays an amount equal to the *second-highest* bid. A strategy for the auction is a **bidding function** β, which is a function of the bidder's valuation. The bidding function determines how much to bid as a function of the bidder's valuation, and the goal is to find a bidding function $\beta(\cdot)$ which maximizes your expected utility (0 if you do not win, and your valuation minus the amount of money you bid if you do win).

(a) For the first-price auction, consider the following scenario: each person draws his/her valuation uniformly from the interval $(0,1)$ (so $f(x)=1$ for $x \in (0,1)$). Suppose that the other bidders bid their own valuations (they use $\beta(x)=x$, the identity bidding function). Consider the case where there is only one other bidder. Your Stanford friend insists that you should always bid $\beta(x)=1$. Your Berkeley friend tells your Stanford friend that it would be better to bid

$$\beta(x) = \frac{x}{2}.$$

Who is correct? [Do not simply compute the expected profit and state that one of the friends has a better bidding function—your job is to prove that your friend's bidding function is optimal.]

(b) Consider the same situation as the previous part, but now assume that there are n other bidders. Your Stanford friend again suggests that $\beta(x)=1$ is the best bid. Your Berkeley friend suggests

$$\beta(x) = \frac{n}{n+1}x.$$

Who is correct this time? [Again, prove that your friend's bidding function is optimal.]

(c) Consider a second-price auction with n bidders where the bidders' valuations are i.i.d. with the exponential density (with parameter λ). Again, they use the identity bidding function, $\beta(x)=x$. What is the distribution of the price P at which the item sells?