UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 5
Fall 2018

Issued: Thursday, September 20, 2018 **Due:** Wednesday, September 26, 2018

Problem 1. **Midterm**
Solve all of the problems on the midterm again (including the ones you got correct).

Problem 2. **Revisiting Facts Using Transforms**

1. Let $X \sim \mathit{Poisson}(\lambda)$, $Y \sim \mathit{Poisson}(\mu)$ be independent. Calculate the MGF of $X + Y$ and use this to show that $X + Y \sim \mathit{Poisson}(\lambda + \mu)$.
2. Calculate the MGF of $X \sim \mathit{Exponential}(\lambda)$ and use this to find all of the moments of X.
3. Repeat the above part, but for $X \sim \mathcal{N}(0, 1)$.

Problem 3. **Almost Sure Convergence**
In this question, we will explore almost sure convergence and compare it to convergence in probability. Recall that a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ converges **almost surely** (abbreviated a.s.) to X if $\Pr(\lim_{n\to\infty} X_n = X) = 1$.

1. Suppose that, with probability 1, the sequence $(X_n)_{n\in\mathbb{N}}$ oscillates between two values $a \neq b$ infinitely often. Is this enough to prove that $(X_n)_{n\in\mathbb{N}}$ does *not* converge almost surely? Justify your answer.
2. Suppose that Y is uniform on $[-1, 1]$, and X_n has distribution

   $$\Pr\big(X_n = (y + n^{-1})^{-1} \bigm| Y = y\big) = 1.$$

   Does $(X_n)_{n=1}^{\infty}$ converge a.s.?
3. Define random variables $(X_n)_{n\in\mathbb{N}}$ in the following way: first, set each X_n to 0. Then, for each $k \in \mathbb{N}$, pick j uniformly randomly in $\{2^k, \dots, 2^{k+1} - 1\}$ and set $X_j = 2^k$. Does the sequence $(X_n)_{n\in\mathbb{N}}$ converge a.s.?
4. Does the sequence $(X_n)_{n\in\mathbb{N}}$ from the previous part converge in probability to some X? If so, is it true that $\mathbb{E}[X_n] \to \mathbb{E}[X]$ as $n \to \infty$?

Problem 4. **Confidence Interval Comparisons**
In order to estimate the probability of a head in a coin flip, p, you flip a coin n times, where n is a positive integer, and count the number of heads, S_n. You use the estimator $\hat{p} = S_n/n$.

(a) You choose the sample size n to have a guarantee

$$\Pr(|\hat{p} - p| \geq \epsilon) \leq \delta.$$

Using Chebyshev Inequality, determine n with the following parameters:

(i) Compare the value of n when $\epsilon = 0.05$, $\delta = 0.1$ to the value of n when $\epsilon = 0.1, \delta = 0.1$.

(ii) Compare the value of n when $\epsilon = 0.1, \delta = 0.05$ to the value of n when $\epsilon = 0.1, \delta = 0.1$.

(b) Now, we change the scenario slightly. You know that $p \in (0.4, 0.6)$ and would now like to determine the smallest n such that

$$\Pr\Big(\frac{|\hat{p} - p|}{p} \leq 0.05\Big) \geq 0.95.$$

Use the CLT to find the value of n that you should use.

Problem 5. **A Chernoff Bound for the Sum of Coin Flips**
Let $X_1, \ldots, X_n$ be i.i.d. *Bernoulli*(q) random variables with bias $q \in (0, 1)$, and call X their sum, $X = X_1 + \cdots + X_n$, which a *Binomial*(n, q) random variable, with mean $\mathbb{E}[X] = nq$.

1. Let $\epsilon > 0$ such that $q + \epsilon < 1$, and define $p = q + \epsilon$. Show that for any $t > 0$,

$$\Pr(X \geq pn) \leq \exp\big(-n(tp - \ln \mathbb{E}[\exp^{tX_1}])\big).$$

2. The *Kullback-Leibler divergence* from the distribution *Bernoulli*(q) to the distribution *Bernoulli*(p), is defined as

$$D(p \parallel q) \triangleq p \ln \frac{p}{q} + (1 - p) \ln \frac{1 - p}{1 - q}.$$

The Kullback-Leibler divergence can be interpreted as a measure of how close the two distributions are. One motivation for this interpretation is that the Kullback-Leibler divergence is always nonnegative, i.e. $D(p \parallel q) \geq 0$, and $D(p \parallel q) = 0$ if and only if $p = q$. So it can be thought of as a 'distance' between the two Bernoulli distributions.

Optimize the previous bound over $t > 0$ and deduce that

$$\Pr(X \geq pn) \leq \exp^{-nD(p \parallel q)}.$$

3. Moreover, the Kullback-Leibler divergence is related to the square distance between the parameters p and q via the following inequality

$$D(p \parallel q) \geq 2(p-q)^2, \quad \text{for } p, q \in (0,1).$$

Use this inequality in order to deduce that

$$\Pr\big(X \geq (q+\epsilon)n\big) \leq \exp^{-2n\epsilon^2},$$

and

$$\Pr\big(X \leq (q-\epsilon)n\big) \leq \exp^{-2n\epsilon^2}.$$

Hint: For the second bound use symmetry in order to avoid doing all the work again.

4. Conclude that

$$\Pr(|X - qn| \geq \epsilon n) \leq 2\exp^{-2n\epsilon^2}.$$

Problem 6. Please complete the feedback form posted in Piazza.