UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 9

Fall 2018

Issued: Thursday, October 18, 2018 **Due:** Wednesday, October 24, 2018

Problem 1. **Sub-Critical Forest**
Consider a random graph $\mathcal{G}(n, p(n))$ where $p(n) \ll 1/n$ (this is called the **sub-critical phase**). Show that the probability that $\mathcal{G}(n, p(n))$ is a forest, i.e. contains no cycles, tends to 1 as $n \to \infty$. [If X_n is the number of cycles, compute $\mathbb{E}[X_n]$ and show that $\mathbb{E}[X_n] \to 0$. Then, apply the First Moment Method, i.e., $\mathbb{P}(X_n > 0) = \mathbb{P}(X_n \geq 1) \leq \mathbb{E}[X_n]$.]

Problem 2. **Poisson Process Warm-Up**
Consider a Poisson process $\{N_t, t \geq 0\}$ with rate $\lambda = 1$. Let random variable S_i denote the time of the ith arrival, where i is a positive integer.

(a) Given $S_3 = s$, where $s > 0$, find the joint distribution of S_1 and S_2.

(b) Find $\mathbb{E}[S_2 \mid S_3 = s]$.

(c) Find $\mathbb{E}[S_3 \mid N_1 = 2]$.

(d) Give an interpretation, in terms of a Poisson process with rate λ, of the following fact:

If N is a geometric random variable with parameter p, and $(X_i)_{i\in\mathbb{N}}$ are i.i.d. exponential random variables with parameter λ, then $X_1 + \cdots + X_N$ has the exponential distribution with parameter λp.

Problem 3. **Expected Squared Arrival Times**
Let $(N(t), t \geq 0)$ be a Poisson process with arrival instants $(T_n, n \in \mathbb{N})$, where $0 < T_1 < T_2 < \cdots$. Find $\mathbb{E}(\sum_{k=1}^{3} T_k^2 \mid N(1) = 3)$.

Problem 4. **Illegal U-Turns**
Each morning, as you pull out of your driveway, you would like to make a U-turn rather than drive around the block. Unfortunately, U-turns are illegal and police cars drive by according to a Poisson process with rate λ. You decide to make a U-turn once you see that the road has been clear of police cars for $\tau > 0$ units of time. Let N be the number of police cars you see before you make a U-turn.

(a) Find $\mathbb{E}[N]$.

(b) Let n be a positive integer ≥ 2. Find the conditional expectation of the time elapsed between police cars $n-1$ and n, given that $N \geq n$.

(c) Find the expected time that you wait until you make a U-turn.

Problem 5. **Spatial Poisson Process**
A two-dimensional Poisson process of rate $\lambda > 0$ is a process of randomly occurring special points in the plane such that (i) for any region of area A the number of special points in that region has a Poisson distribution with mean λA, and (ii) the number of special points in non-overlapping regions is independent. For such a process consider an arbitrary location in the plane and let X denote its distance from its nearest special point (where distance between two points (x_1, y_1) and (x_2, y_2) is defined as $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$). Show that:

(a) $\mathbb{P}(X > t) = \exp(-\lambda \pi t^2)$ for $t > 0$.

(b) $\mathbb{E}[X] = \dfrac{1}{2\sqrt{\lambda}}$.

Problem 6. **Bus Arrivals at Cory Hall**
Starting at time 0, the 52 line makes stops at Cory Hall according to a Poisson process of rate λ. Students arrive at the stop according to an independent Poisson process of rate μ. Every time the bus arrives, all students waiting get on.

1. Given that the interarrival time between bus $i-1$ and bus i is x, find the distribution for the number of students entering the ith bus. (Here, x is a given number, not a random quantity.)

2. Given that a bus arrived at 9:30 AM, find the distribution for the number of students that will get on the next bus.

3. Find the distribution of the number of students getting on the next bus to arrive after 9:30 AM, assuming that time 0 was infinitely far in the past.

Problem 7. **Bonus: Sum-Quota Sampling**
Consider the problem of estimating the mean interarrival time of a Poisson process. In what follows, recall that N_t denotes the number of arrivals by time t, where $t \geq 0$. *Sum-quota sampling* is a form of sampling in which the number of samples is not fixed in advance; instead, we wait until a fixed *time* t, and take the average of the interarrival times seen so far. If we let X_i denote the ith interarrival time, then

$$\bar{X} = \frac{X_1 + \cdots + X_{N_t}}{N_t}.$$

Of course, the above quantity is not defined when $N_t = 0$, so instead we must condition on the event $\{N_t > 0\}$. Compute $\mathbb{E}[\bar{X} \mid N_t > 0]$, assuming that $(N_t,\ t \geq 0)$ is a Poisson process of rate λ.