

STATE-SPACE EQ: $x_n = a x_{n-1} + v_{n-1}$
 $y_n = c x_n + w_n$

Goal: Est. X_n given $Y^n = \{y_1, \dots, y_n\}$ in an online fashion.

Notation: $L[X_n | Y^n] = \hat{x}_{n|n}$; $L[X_n | Y^{n-1}] = \hat{x}_{n|n-1}$
 $E[(\hat{x}_n - \hat{x}_{n|n})^2] = \sigma_{n|n}^2$; $E[(\hat{x}_n - \hat{x}_{n|n-1})^2] = \sigma_{n|n-1}^2$
 $E[v_n^2] = \sigma_v^2$; $E[w_n^2] = \sigma_w^2$

Kalman Eqs. (Scalar Case)

① $\hat{x}_{n|n} = \hat{x}_{n|n-1} + K_n \tilde{y}_n$ (innovations)
(a) $\hat{x}_{n|n-1} = a \hat{x}_{n-1|n-1}$; (b) $\tilde{y}_n = y_n - L[y_n | Y^{n-1}]$
 $= y_n - \hat{x}_{n|n-1}$

② $K_n = \frac{c \sigma_{n|n-1}^2}{c^2 \sigma_{n|n-1}^2 + \sigma_w^2}$

③ $\sigma_{n|n-1}^2 = a^2 \sigma_{n-1|n-1}^2 + \sigma_v^2$

④ $\sigma_{n|n}^2 = \sigma_{n|n-1}^2 (1 - K_n)$

$$x_n = a x_{n-1} + v_n$$

$$y_n = x_n + w_n$$

(c=1)

can be
pre-computed

$$\textcircled{1} \hat{x}_{n|n} = \hat{x}_{n|n-1} + k_n \tilde{y}_n$$

$$\begin{aligned} \cdot \tilde{y}_n &= y_n - a \hat{x}_{n-1|n-1} \\ \cdot \hat{x}_{n|n-1} &= a \hat{x}_{n-1|n-1} \end{aligned}$$

$$\textcircled{2} k_n = \frac{\sigma_{n|n-1}^2}{\sigma_{n|n-1}^2 + \sigma_w^2}$$

$$\textcircled{3} \sigma_{n|n-1}^2 = a^2 \sigma_{n-1|n-1}^2 + \sigma_v^2$$

$$\textcircled{4} \sigma_{n|n}^2 = \sigma_{n|n-1}^2 (1 - k_n)$$

KALMAN

EQS. (c=1)

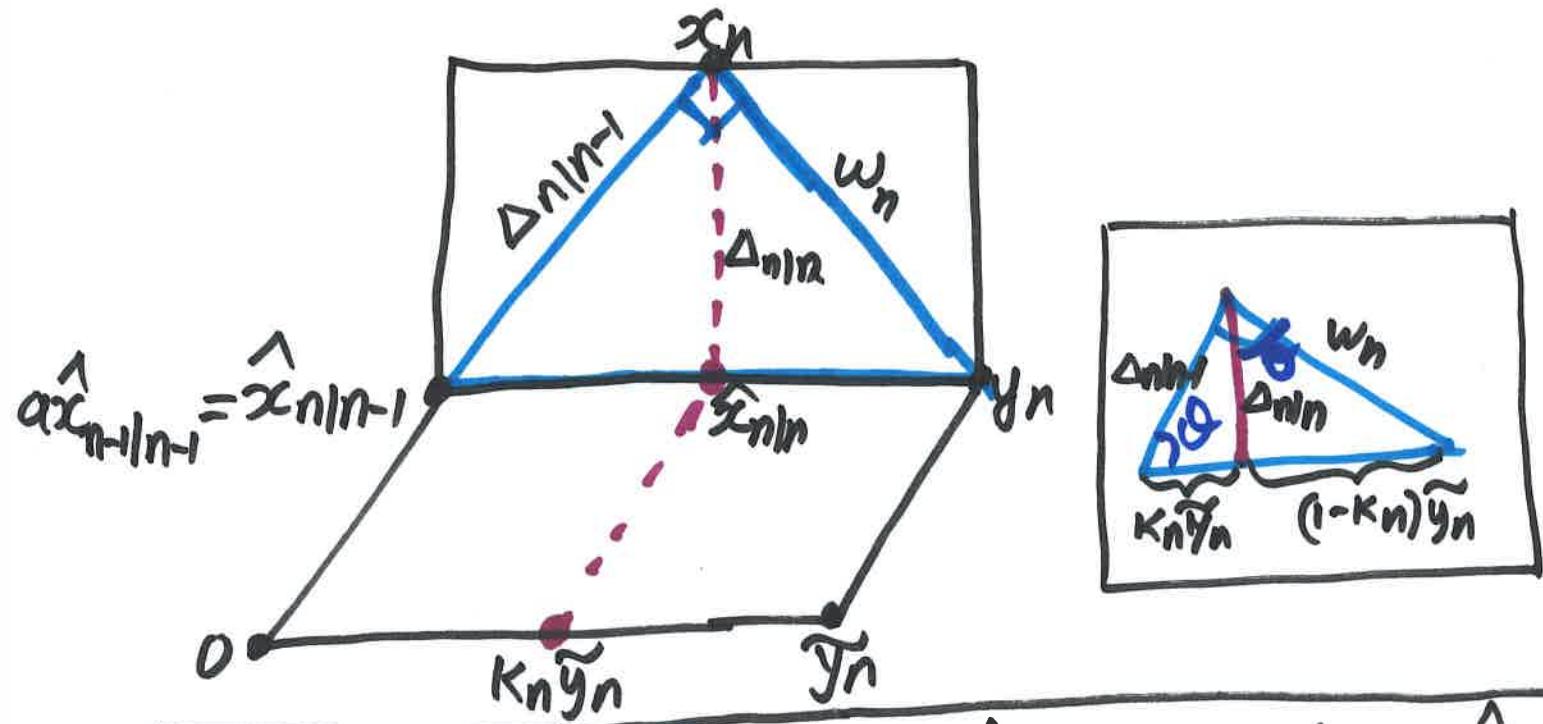
Remarks:

(1) At iteration n , the algorithm inputs $\hat{x}_{n-1|n-1}$, $\sigma_{n-1|n-1}^2$ (and new obs. y_n), and outputs $\hat{x}_{n|n}$, $\sigma_{n|n}^2$ for the next iteration.

(2) The Kalman gain k_n and the errors $\sigma_{n|n-1}^2$, $\sigma_{n|n}^2$ can be pre-computed.
(Only $\hat{x}_{n|n}$ needs to be computed online)

(3) Easy to implement: only a few lines of code!

(4) If $v_n, w_n \sim N(\cdot)$, Kalman Filter \Rightarrow MMSE est.



$$① \hat{x}_{n|n} = \hat{x}_{n|n-1} + K_n \tilde{y}_n = a \hat{x}_{n|n-1} + K_n (y_n - a \hat{x}_{n|n-1}) \quad \text{by inspection.}$$

② Finding K_n : Δ 's $(\hat{x}_{n|n-1}, x_n, y_n)$ & $(\hat{x}_{n|n-1}, \hat{x}_{n|n}, x_n)$ are similar.

$$\frac{\|K_n \tilde{y}_n\|}{\|\Delta_{n|n-1}\|} = \frac{\|\Delta_{n|n-1}\|}{\|\tilde{y}_n\|} \Rightarrow K_n = \frac{\|\Delta_{n|n-1}\|^2}{\|\tilde{y}_n\|^2} = \frac{\sigma_{n|n-1}^2}{\sigma_{n|n-1}^2 + \sigma_w^2}$$

$$③ \text{Finding } \sigma_{n|n}^2: (a) \frac{\|(1-K_n) \tilde{y}_n\|}{\|w_n\|} = \frac{\|w_n\|}{\|\tilde{y}_n\|} \Rightarrow 1-K_n = \frac{\|w_n\|^2}{\|\tilde{y}_n\|^2}$$

$$(b) \frac{\|\Delta_{n|n}\|}{\|\Delta_{n|n-1}\|} = \frac{\|w_n\|}{\|\tilde{y}_n\|} \Rightarrow \|\Delta_{n|n}\| = \|\Delta_{n|n-1}\| \cdot \frac{\|w_n\|}{\|\tilde{y}_n\|} \Rightarrow \sigma_{n|n}^2 = \sigma_{n|n-1}^2 (1-K_n)$$

$$④ \bar{\sigma}_{n|n-1}^2 = \overline{\mathbb{E}[(\bar{x}_n - \hat{x}_{n|n-1})^2]} = \overline{\mathbb{E}[(a x_{n-1} + v_n - a \hat{x}_{n|n-1})^2]} = \overline{\mathbb{E}[(a \Delta_{n-1|n-1} + v_n)^2]} = a^2 \sigma_{n-1|n-1}^2 + \sigma_v^2$$

VECTOR CASE

$$X_n = A X_{n-1} + V_n$$

$$Y_n = C X_n + W_n$$

Kalman Eqs.

$$\textcircled{1} \quad \hat{X}_{n|n} = \hat{X}_{n|n-1} + K_n \tilde{Y}_n$$

$$(a) \hat{X}_{n|n-1} = A \hat{X}_{n-1|n-1} \quad ; \quad (b) \tilde{Y}_n = Y_n - C \hat{X}_{n|n-1}$$

$$\textcircled{2} \quad K_n = \Sigma_{n-1|n-1} C^T (C \Sigma_{n|n-1} C^T + \Sigma_w)^{-1}$$

$$\textcircled{3} \quad \Sigma_{n|n-1} = A \Sigma_{n-1|n-1} A^T + \Sigma_v$$

$$\textcircled{4} \quad \Sigma_{n|n} = (I - K_n C) \Sigma_{n|n-1}$$