Name: SOLUTIONS SID:

(24 points total)

Problem 1: A student wants to analyze to probability of drawing certain hands in poker. In this problem we will consider a deck of 52 cards consisting of four suits of each of the numbers 1-13. A hand will consist of 5 cards.

Note: when talking about a hand, the order of the cards does not matter. So: {1,2,2,4,5} and {2,4,1,5,2} are equivalent.

(6 points)

1.a) What is the probability of a hand having exactly three "1"s?

$$\frac{\binom{4}{3}\cdot\binom{48}{2}}{\binom{52}{5}}$$

THERE ARE 4 ONES, AND YOU MUST CHOOSE 3 FOR YOUR HAND

THERE ARE 52 CHOOSE 5 TOTAL POSSIBLE HANDS.

YOU CAN THEN CHOOSE ANY TWO OF THE REMAINING 48 CARDS FOR THE REST OF THE HAND

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(6 points)
1.b) What is the probability of getting a straight? This means the hand contains cards that ascend consecutively. For example, a hand containing the numbers {4,5,6,7,8}, or the numbers {2,3,4,5,6}, etc. Note that in this problem, card 1 is NOT greater than card 13 as is sometimes considered in poker, so {10,11,12,13,1} is not a straight.

$$\frac{9 \cdot 4^5}{\binom{52}{5}}$$

EACH CARD IN THE STRAIGHT CAN BE FROM ANY ONE OF THE 4 SUITS.

THERE ARE 9 POSSIBLE STRAIGHTS:

[1,5]
[2,6]
[3,7]
[4,8]
[5,9]
[6,10]
[7,11]
[8,12]
[9,13]

THERE ARE 52 CHOOSE 5 POSSIBLE HANDS

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(6 points)
1.c) What is the probability of getting a straight, drawn temporally in ascending order? That is, you draw the lowest card first, followed by the second lowest, followed by third lowest, followed by forth lowest, followed by the highest card.

$$\frac{9 \cdot 4^5}{\binom{52}{5} \cdot 5!}$$

THE NUMBER OF CORRECT SOLUTIONS HASNT CHANGED

BUT THE NUMBER OF TOTAL SOLUTIONS HAS AS WE NOW CONSIDER ALL POSSIBLE ORDERINGS.

NOTE: THIS IS EQUIVALENT TO 52p5

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(6 points)

1.d) What is the probability of getting a hand containing three of any one number, and two of any other number? Examples: {3,3,3,2,2}, {5,5,5,9,9}, {12,12,12,6,6}

SECOND NUMBER CAN BE ANY OF THE 12 REMAINING NUMBERS

$$\frac{13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}}{\binom{52}{5}}$$

FIRST NUMBER CAN BE ANY OF THE 13

WE CHOOSE 2 OF THE 4 CARDS OF THAT NUMBER

WE CHOOSE 3 OF THE 4 COPIES OF THAT NUMBER

THERE ARE 52 CHOOSE 5 TOTAL POSSIBLE HANDS

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(36 points total)

Problem 2: To the best of our knowledge, with probability 0.8 Al is guilty of the crime for which he is about to be tried. Bo and Ci, each of whom knows whether or not Al is guilty, have been called to testify.

Bo is a friend of Al's and will tell the truth if Al is innocent, but will lie with probability 0.2 if Al is guilty. Ci hates everybody but the judge and will tell the truth if Al is guilty but will lie with probability 0.3 if Al is innocent.

Given this model of the physical situation, answer the following questions.

EVENT A: AL IS INNOCENT

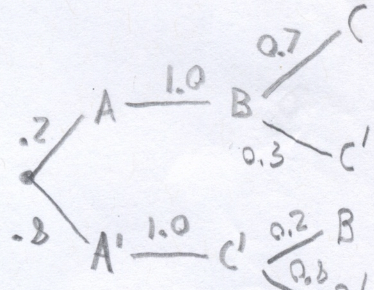
EVENT B: BO TESTIFIES THAT AL IS INNOCENT

EVENT C: CI TESTIFIES THAT AL IS INNOCENT

EVENT X: THE WITNESSES GIVE CONFLICTING TESTIMONY

EVENT Y: BO COMMITS PERJURY

EVENT Z: CI COMMITS PERJURY



SAMPLE POINTS

$P(ABC) = .14$

$P(ABC') = .06$

$P(A'BC') = .16$

$P(A'B'C') = .64$

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(10 points)

2.a) Determine the probability that the witnesses give conflicting testimony.

$$P(X) = P(BC' + B'C) = P(BC') + \cancel{P(B'C)}^{\,0}$$

$$= P(ABC') + P(A'BC') \quad \text{BY LAW OF TOTAL PROBABILITY}$$

$$= 0.16 + 0.06$$

$$\boxed{P(X) = 0.22}$$

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(8 points)

2.b) Which witness is more likely to commit perjury (lie)?

$$P(Y) = P(AB' + A'B) = P(AB') + P(A'B)$$

$$= 0.00 + 0.16$$

$$\boxed{P(Y) = 0.16}$$

$$P(Z) = P(AC' + A'C) = P(AC') + P(A'C)$$

$$= 0.06 + 0.00$$

$$= 0.06$$

$$\boxed{\text{BO IS MORE LIKELY TO LIE}}$$

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(10 points)
2.c) What is the conditional probability that Al is innocent, given that Bo and Ci gave conflicting testimony?

$$P(A|X) = \frac{P(A,X)}{P(X)} = \frac{P(A,B,C') + P(\overset{0}{\cancel{A}}, B', C)}{P(X)}$$

$$= \frac{P(A,B,C')}{P(X)} = \frac{0.06}{0.22} = \frac{3}{11}$$

$$\boxed{P(A|X) = 3/11}$$

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(8 points)
2.d) Are the events "Bo tells a lie" and "Ci tells a lie" independent? Are these events conditionally independent to an observer who knows whether or not Al is guilty?

TO CHECK FOR INDEPENDENCE WE TEST

$$P(YZ) = P(Y)P(Z)$$

WE CAN SEE FROM OUR MODEL THAT BO AND CI NEVER <u>BOTH</u> LIE, SO $P(YZ)=0$

BUT WE ALREADY SHOWED $P(Y)$ AND $P(Z)$ ARE NON-ZERO.

THEREFORE:

$$P(YZ) \neq P(Y)P(Z)$$

$\boxed{\text{• THE EVENTS ARE NOT INDEPENDENT.}}$

TO TEST FOR CONDITIONAL INDEPENDENCE WE NEED TO TEST

$$\underbrace{P(YZ|A)}_{\text{STILL } 0} = P(Y|A)\underbrace{P(Z|A)}_{0}$$

$0 = 0$ ☑

AND TEST:

$$\underbrace{P(YZ|A')}_{\text{STILL } 0} = P(Y|A')\underbrace{P(Z|A')}_{0}$$

$0 = 0$ ☑

$\boxed{\text{• THE EVENTS ARE CONDITIONALLY INDEPENDENT TO AN OBSERVER WHO KNOWS WHETHER OR NOT AL IS GUILTY}}$

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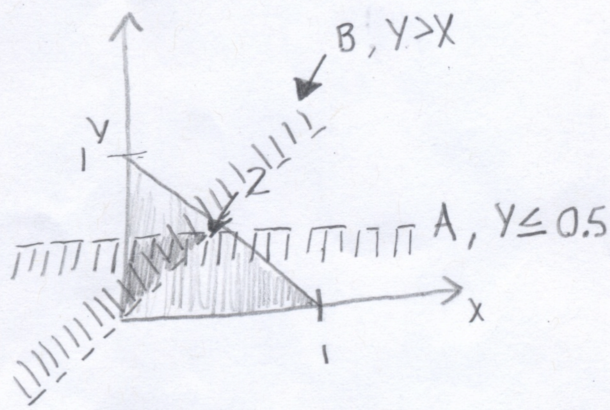
(40 points total)
Problem 3: Random variables X and Y have the joint PDF:

$$f_{X,Y}(x_0, y_0) = \begin{cases} 2 & \text{if } x_0 > 0 \text{ and } y_0 > 0 \text{ and } x_0 + y_0 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Let A be the event $Y \leq 0.5$.
Let B be the event $Y > X$.

(10 points)
3.a) Determine the numerical value of $P(B|A)$



- THE DARKER SHADED REGION IS THE AREA FOR WHICH BOTH B, AND A ARE TRUE
- SINCE $f_{XY}(X_0, Y_0)$ IS UNIFORM, THE PROBILITY OF A REGION IS PROPORTIONAL TO THE AREA OF THAT REGION

$$P(B|A) = \frac{P(B,A)}{P(A)} = \frac{1/8 \cdot 2}{3/8 \cdot 2} = 1/3$$

$$\boxed{P(B|A) = 1/3}$$

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(40 points total)

Problem 3: Random variables X and Y have the joint PDF:

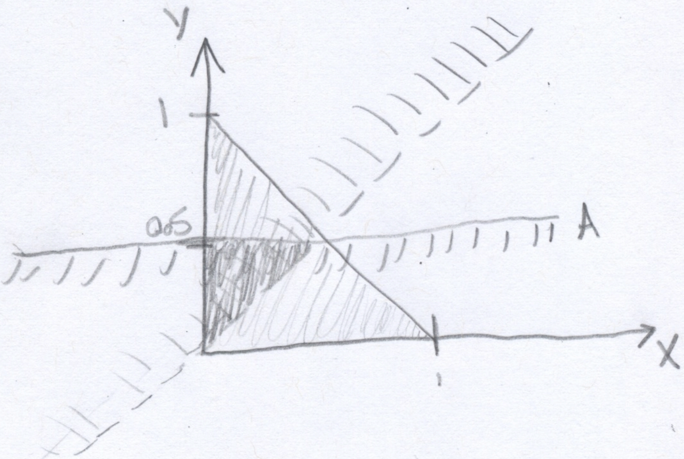
$$f_{X,Y}(x_0, y_0) = \begin{cases} 2 & \textit{if } x_0 > 0 \textit{ and } y_0 > 0 \textit{ and } x_0 + y_0 \leq 1 \\ 0 & \textit{otherwise} \end{cases}$$

Let A be the event $Y \leq 0.5$.
Let B be the event $Y > X$.

(10 points)

3.a) Determine the numerical value of $P(B|A)$

LONG WAY



$$P(B|A) = \frac{P(B,A)}{P(A)} = \frac{\int_0^{0.5}\int_X^{0.5} 2\, dy\,dx}{\int_0^{0.5}\int_0^{1-y} 2\, dx\,dy} = \frac{\int_0^{0.5} 1-2x}{\int_0^{0.5} 2-2y} = \frac{X-X^2\Big|_0^{0.5}}{2y-y^2\Big|_0^{0.5}} = \frac{1/4}{3/4} =$$

$P(B|A) = 1/3$

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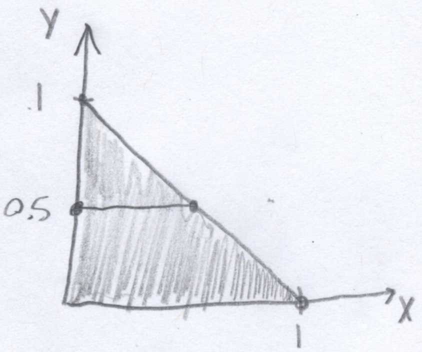
(10 points)
3.b) Prepare a neat, fully labeled sketch of $f_{X|Y}(x_0|0.5)$. Also evaluate the conditional expectation and the conditional variance for X, given that the experimental value of Y is equal to 0.5.

$f(x,y) =$

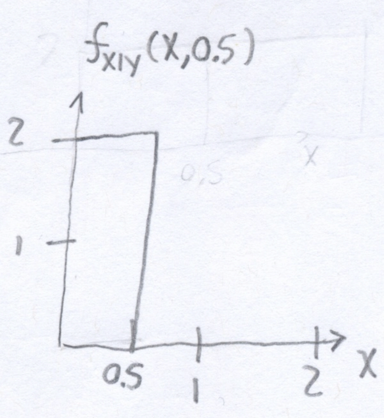


- X WILL BE UNIFORM ON THE RANGE [0, 0.5]

$$f_{X|Y}(X|0.5) = \frac{f_{X,Y}(X, 0.5)}{f_Y(0.5)}$$

$f_{X,Y}(X, 0.5)$ ← 2 FOR $0 < X < 0.5$, 0 OTHERWISE

$f_Y(0.5)$ ← CONSTANT



$$\boxed{E[X|0.5] = \underbrace{\frac{0.5+0}{2}}_{\text{MEAN OF A UNIFORM DISTRIBUTION}} = 0.25}$$

$$\boxed{VAR[X|0.5] = \underbrace{\frac{(0.5-0)^2}{12}}_{\text{VARIANCE OF A UNIFORM DISTRIBUTION}} = \frac{0.25}{12} = \frac{1}{48}}$$

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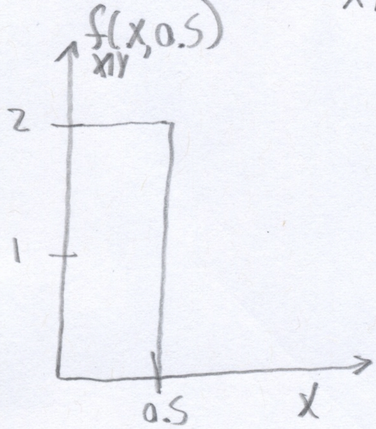
(10 points)
3.b) Prepare a neat, fully labeled sketch of $f_{X|Y}(x_0|0.5)$. Also evaluate the conditional expectation and the conditional variance for X, given that the experimental value of Y is equal to 0.5.

LONG WAY

$$f_{X|Y}(x_0|0.5) = \frac{f_{X,Y}(x_0, 0.5)}{f_Y(0.5)}$$

$f_{X,Y}(x_0, 0.5)$: 2 for $0 \le x \le 0.5$, 0 OTHERWISE

$f_Y(0.5) \leftarrow \int_x f_{X,Y}(x, 0.5)$



$$f_Y(0.5) = \int_0^{0.5} 2\, dx$$

$$f_Y(0.5) = 1$$

$$\boxed{f_{X|Y}(x_0|0.5) = \begin{cases} 2 & \text{for } 0 \le x \le 0.5 \\ 0 & \text{OTHERWISE} \end{cases}}$$

$$E[X|Y=0.5] = \int_0^{0.5} 2x = x^2 \Big|_0^{0.5} = .25$$

$$\boxed{E[X|Y=0.5] = 0.25}$$

$$E[X^2|Y=0.5] = \int_0^{0.5} 2x^2 = \frac{2}{3}x^3 \Big|_0^{0.5} = \frac{1}{12}$$

$$VAR[X|Y=0.5] = E[(X - E[X|Y=0.5])^2 | Y=0.5] = E[X^2|Y=0.5] - E[X|Y=0.5]^2$$

$$= \frac{1}{12} - \frac{1}{16} = \frac{1}{48}$$

$$\boxed{VAR[X|Y=0.5] = 1/48}$$

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(10 points)

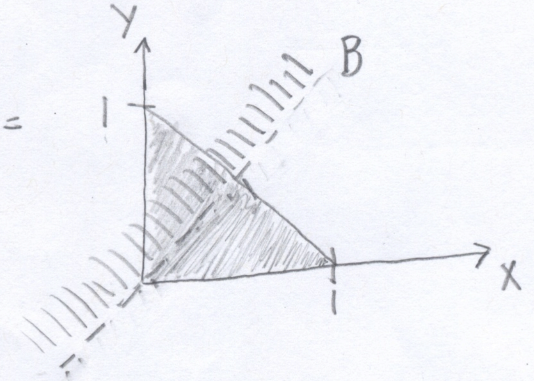
3.c) Prepare a neat, fully labeled sketch of $f_{X|B}(x_0|B)$.

• for X_0, Y_0 in B

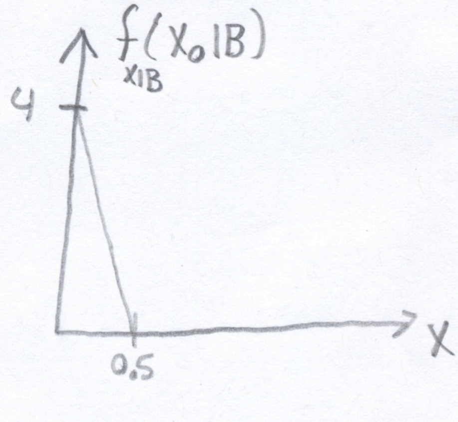
FOR $0 < X < 0.5$

$$f_{X|B}(x_0|B) = \int_y \frac{f_{X,Y}(x_0,y_0)}{P(B)} = \frac{\int_x^{1-x} 2}{\frac{1}{4} \cdot 2} = \frac{2-4x}{1/2} = 4-8x$$

$f_{XY}(x,y) =$



$f_{X|B}(x_0|B) =$



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(10 points)

3.d) Let T = XY. Determine the numerical value of E[T].

$$E[T] = E[XY] = \iint_{x_0 y_0} x_0 y_0 \, f_{X,Y}(x_0, y_0)\, dy_0\, dx_0$$

$$= \int_0^1 \int_0^{1-x_0} x_0 y_0 \cdot 2 \, dy_0 dx_0$$

$$= \int_0^1 x_0 \cdot (1-x_0)^2$$

$$= \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \Big|_0^1$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4}$$

$$\boxed{E[T] = \frac{1}{12}}$$

(END OF EXAM) Please make sure your name and SID are at the top of every page.