UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 1
Fall 2019

Issued: August 28, 2019 **Due:** 11:59 PM, Wednesday, September 4, 2019

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1. **Miscellaneous Review**

   (a) Show that the probability that exactly one of the events A and B occurs is $\mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B)$.

   (b) If A is independent of itself, show that $\mathbb{P}(A) = 0$ or 1.

2. **Coin Flipping & Symmetry**

   Alice and Bob have $2n+1$ fair coins (where $n \in \mathbb{Z}_{>0}$), each coin with probability of heads equal to 1/2. Bob tosses $n+1$ coins, while Alice tosses the remaining n coins. Assuming independent coin tosses, show that the probability that, after all coins have been tossed, Bob will have gotten more heads than Alice is 1/2.

   Hint: This problem has a two-line solution.

3. **Choosing from Any Jar Makes No Difference**

   Each of k jars contains w white and b black balls. A ball is randomly chosen from jar 1 and transferred to jar 2, then a ball is randomly chosen from jar 2 and transferred to jar 3, etc. Finally, a ball is randomly chosen from jar k. Show that the probability that the last ball is white is the same as the probability that the first ball is white, i.e., it is $w/(w+b)$.

4. **Passengers on a Plane**

   There are N passengers in a plane with N assigned seats (N is a positive integer), but after boarding, the passengers take the seats randomly. Assuming all seating arrangements are equally likely, what is the probability that no passenger is in their assigned seat? Compute the probability when $N \to \infty$.

   Hint: Use the inclusion-exclusion principle

5. **Expanding the NBA**

   The NBA is looking to expand to another city. In order to decide which city will receive a new team, the commissioner interviews potential owners from each of the N potential cities (N is a positive integer), one at a time. Unfortunately, the owners would like to know immediately after the interview whether their city will receive the team or not. The commissioner decides to use the following strategy: she will interview the first m owners and reject all of them ($m \in \{1, \dots, N\}$). After the mth owner is interviewed, she will pick the first city that is better than all previous cities. The cities are interviewed in a uniformly random order. What is the probability that the best city is selected? Assume that the commissioner has an objective method of scoring each city and that each city receives a unique score.

You should arrive at an exact answer for the probability in terms of a summation. Approximate your answer using $\sum_{i=1}^{n} i^{-1} \approx \ln n$ and find the optimal value of m that maximizes the probability that the best city is selected.

6. **Superhero Basketball**

   Superman and Captain America are playing a game of basketball. At the end of the game, Captain America scored n points and Superman scored m points, where $n > m$ are positive integers. Supposing that each basket counts for exactly one point, what is the probability that after the start of the game (when they are initially tied), Captain America was always *strictly* ahead of Superman? (Assume that all sequences of baskets which result in the final score of n baskets for Captain America and m baskets for Superman are equally likely.)

   Hint: Think about symmetry. First, try to figure out which is more likely: there was a tie and Superman scored the first point, or there was a tie and Captain America scored the first point?

7. **[Bonus] Tournament Probabilistic Proof**

   The bonus question is just for fun. You are not required to submit the bonus question, but do give it a try and write down your progress.

   In a *tournament* with n players (where n is a positive integer), each player plays against every other player for a total of $\binom{n}{2}$ games (assume that there are no ties). Let k be a positive integer. Is it always possible to find a tournament such that for any subset A of k players, there is a player who has beaten everyone in A? For such a tournament, let us say that every k-subset is dominated. For example, Figure 1 depicts the smallest tournament in which every 2-subset is dominated.

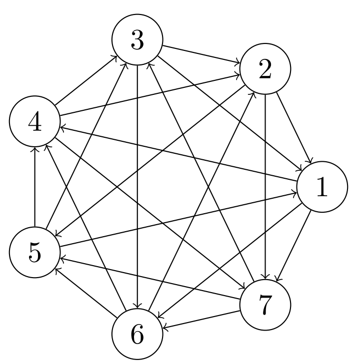

Figure 1: A tournament with 7 vertices such that every pair of players is beaten by a third player.

In fact, as long as $\binom{n}{k}(1-2^{-k})^{n-k} < 1$, it is possible to find a tournament of n players such that every k-subset is dominated. Prove this fact, and explain why it implies that for any positive integer k there exists a tournament such that every k-subset is dominated.