UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 11
Fall 2019

1. **Midterm**

   Solve all of the problems on the midterm again (including the ones you got correct).

2. **Random Bipartite Graph**

   Consider a random bipartite graph with, K left nodes and M right nodes. Each of the $K \cdot M$ possible edges of this graph is present with probability p independently.

   (a) Find the distribution of the degree of a particular right node.

   (b) Now suppose you pick an edge which is present in the graph. What is the distribution of the degree of the right node connected to this edge? Is it the same as in part (a)?

   (c) We call a right node with degree one a *singleton*. What is the average number of singletons in a random bipartite graph?

   (d) Find the average number of left nodes that are connected to at least one singleton.

3. **Isolated Vertices**

   Consider a Erdös-Renyi random graph $\mathcal{G}(n, p(n))$, where n is the number of vertices and $p(n)$ is the probability that a specific edge appears in the graph. Let X_n be the number of isolated vertices in $\mathcal{G}(n, p(n))$. Show that

$$\mathbb{E}[X_n] \xrightarrow{n\to\infty} \begin{cases} \infty, & p(n) \ll \dfrac{\ln n}{n}, \\ \exp(-c), & p(n) = \dfrac{\ln n + c}{n}, \\ 0, & p(n) \gg \dfrac{\ln n}{n}, \end{cases}$$

   where the notation $p(n) \ll f(n)$ means that $p(n)/f(n) \to 0$ as $n \to \infty$, and $p(n) \gg f(n)$ means $p(n)/f(n) \to \infty$ as $n \to \infty$. Show also that in the third case, $p(n) \gg (\ln n)/n$, we have $X_n \to 0$ in probability as well.

4. **BSC: MLE & MAP**

You are testing a digital link that corresponds to a BSC with some error probability $\epsilon \in [0, 0.5]$.

(a) Assume you observe the input and the output of the link. How do you find the MLE of ϵ?

(b) You are told that the inputs are i.i.d. bits that are equal to 1 with probability 0.6 and to 0 with probability 0.4. You observe n outputs (n is a positive integer). How do you calculate the MLE of ϵ?

(c) The situation is as in the previous case, but you are told that ϵ has PDF $4 - 8x$ on $[0, 0.5)$. How do you calculate the MAP of ϵ given n outputs?

5. **Random Graph Estimation**

Consider a random graph on n vertices in which each edge appears independently with probability p. Let D be the average degree of a vertex in the graph.

(a) Compute the maximum likelihood estimator of p given D. You may approximate $\text{Binomial}(n, p) = \text{Poisson}(np)$.

(b) Using the CLT, find an approximate 95% confidence interval for p using the maximum likelihood estimator $\hat{p}$ from the previous part. Your answer should not include p.