UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 13
Fall 2019

1. **Geometric MMSE**

Let N be a geometric random variable with parameter $1-p$, and $(X_i)_{i\in\mathbb{N}}$ be i.i.d. exponential random variables with parameter λ. Let $T = X_1 + \cdots + X_N$. Compute the LLSE and MMSE of N given T.

Hint: Compute the MMSE first.

2. **Property of MMSE**

Let $X, Y_1, \dots, Y_n$ be square integrable random variables. Argue that

$$\mathbb{E}\big[(X - \mathbb{E}[X \mid Y_1, \dots, Y_n])^2\big] \le \mathbb{E}\left[\left(X - \sum_{i=1}^{n} \mathbb{E}[X \mid Y_i]\right)^2\right].$$

3. **Gaussian Random Vector MMSE**

Let

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$$

be a Gaussian random vector.

Let

$$W = \begin{cases} 1, & \text{if } Y > 0 \\ 0, & \text{if } Y = 0 \\ -1, & \text{if } Y < 0 \end{cases}$$

be the sign of Y. Find $\mathbb{E}[WX \mid Y]$.

4. **MMSE for Jointly Gaussian Random Variables**

Provide justification for each of the following steps (1 - 5) to prove that the LLSE is equal to the MMSE estimator for jointly Gaussian random variables X and Y.
Let $g(X) = L[Y \mid X]$.

$$E[(Y - g(X))X] = 0 \qquad (1)$$

$$\begin{aligned}
&\implies \operatorname{cov}(Y - g(X), X) = 0 && (2)\\
&\implies Y - g(X) \text{ is independent of } X && (3)\\
&\implies E[(Y - g(X))f(X)] = 0 \ \forall f(\cdot) && (4)\\
&\implies g(X) = E[Y \mid X] && (5)
\end{aligned}$$

5. **Stochastic Linear System MMSE**

Let $(V_n,\ n \in \mathbb{N})$ be i.i.d. $\mathcal{N}(0, \sigma^2)$ and independent of $X_0 = \mathcal{N}(0, u^2)$. Let $|a| < 1$. Define

$$X_{n+1} = aX_n + V_n, \qquad n \in \mathbb{N}.$$

(a) What is the distribution of X_n, where n is a positive integer?

(b) Find $\mathbb{E}[X_{n+m} \mid X_n]$ for $m, n \in \mathbb{N}$, $m \geq 1$.

(c) Find u so that the distribution of X_n is the same for all $n \in \mathbb{N}$.

6. **Random Walk with Unknown Drift**

Consider a random walk with unknown drift. The dynamics are given, for $n \in \mathbb{N}$, as

$$\begin{aligned}
X_1(n+1) &= X_1(n) + X_2(n) + V(n),\\
X_2(n+1) &= X_2(n),\\
Y(n) &= X_1(n) + W(n).
\end{aligned}$$

Here, X_1 represents the position of the particle and X_2 represents the velocity of the particle (which is unknown but constant throughout time). Y is the observation. V and W are independent Gaussian noise variables with mean zero and variance σ_V^2 and σ_W^2 respectively.

(a) Write down the dynamics of the system in matrix-vector form and write down the Kalman filter recursive equations for this system.

(b) Let k be a positive integer. Compute the prediction $\mathbb{E}(X(n+k) \mid Y^{(n)})$, where $Y^{(n)}$ is the history of the observations $Y_0, \ldots, Y_n$, in terms of the estimate $\hat{X}(n) := \mathbb{E}(X(n) \mid Y^{(n)})$.

(c) Now let $k = 1$ and compute the smoothing estimate $\mathbb{E}(X(n) \mid Y^{(n+1)})$ in terms of the quantities that appear in the Kalman filter equation.

Hint: Use the law of total expectation

$$\mathbb{E}\big(X(n) \mid Y^{(n+1)}\big) = \mathbb{E}\big[\mathbb{E}\big(X(n) \mid X(n+1), Y^{(n+1)}\big) \mid Y^{(n+1)}\big],$$

as well as the *innovation*

$$\tilde{X}(n+1) := X(n+1) - L[X(n+1) \mid Y^{(n)}].$$