UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 4 (optional)
Spring 2019

Issued: September 19, 2019 **Due:** No due date

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1. **Transform Practice**

Consider a random variable Z with transform

$$M_Z(s) = \frac{a - 3s}{s^2 - 6s + 8}, \qquad \text{for } |s| < 2.$$

Calculate the following quantities:

(a) The numerical value of the parameter a.
(b) $\mathbb{E}[Z]$.
(c) $\text{var}(Z)$.

2. **Bounds for the Coupon Collector's Problem**

Recall the coupon collector's problem, where X is a random variable which is equal to the number of boxes bought until one of every type of coupon is obtained (there are n total coupons).

The expected value of X is nH_n, where H_n is the *harmonic number of order* n which is defined as

$$H_n \triangleq \sum_{i=1}^{n} \frac{1}{i},$$

and satisfies the inequalities

$$\ln n \leq H_n \leq \ln n + 1.$$

(a) Use Markov's inequality in order to show that

$$\mathbb{P}(X > 2nH_n) \leq \frac{1}{2}.$$

(b) Use Chebyshev's inequality in order to show that

$$\mathbb{P}(X > 2nH_n) \leq \frac{\pi^2}{6(\ln n)^2}.$$

Note: You can use the identity

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}.$$

(c) Define appropriate events and use the union bound in order to show that

$$\mathbb{P}(X > 2nH_n) \leq \frac{1}{n}.$$

Note: The sequence $a_n = (1 - 1/n)^n$, for $n = 1, 2, 3, \ldots$, is strictly increasing and $\lim_{n\to\infty} a_n = 1/\mathrm{e}$.

3. A Chernoff Bound for the Sum of Coin Flips

Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli(q) random variables with bias $q \in (0, 1)$, and call X their sum, $X = X_1 + \cdots + X_n$, which a Binomial(n, q) random variable, with mean $\mathbb{E}[X] = nq$.

(a) Let $\epsilon > 0$ such that $q + \epsilon < 1$, and define $p = q + \epsilon$. Show that for any $t > 0$,

$$\mathbb{P}(X \geq pn) \leq \exp\big(-n(tp - \ln \mathbb{E}[\mathrm{e}^{tX_1}])\big).$$

(b) The *Kullback-Leibler divergence* from the distribution Bernoulli(q) to the distribution Bernoulli(p), is defined as

$$D(p \parallel q) \triangleq p \ln \frac{p}{q} + (1 - p) \ln \frac{1 - p}{1 - q}.$$

The Kullback-Leibler divergence can be interpreted as a measure of how close the two distributions are. One motivation for this interpretation is that the Kullback-Leibler divergence is always nonnegative, i.e. $D(p \parallel q) \geq 0$, and $D(p \parallel q) = 0$ if and only if $p = q$. So it can be thought of as a 'distance' between the two Bernoulli distributions.

Optimize the previous bound over $t > 0$ and deduce that

$$\mathbb{P}(X \geq pn) \leq \mathrm{e}^{-nD(p \parallel q)}.$$

(c) Moreover, the Kullback-Leibler divergence is related to the square distance between the parameters p and q via the following inequality

$$D(p \parallel q) \geq 2(p - q)^2, \quad \text{for } p, q \in (0, 1).$$

Use this inequality in order to deduce that

$$\mathbb{P}\big(X \geq (q + \epsilon)n\big) \leq \mathrm{e}^{-2n\epsilon^2},$$

and

$$\mathbb{P}\big(X \leq (q - \epsilon)n\big) \leq \mathrm{e}^{-2n\epsilon^2}.$$

Hint: For the second bound use symmetry in order to avoid doing all the work again.

(d) Conclude that

$$\mathbb{P}(|X - qn| \geq \epsilon n) \leq 2\mathrm{e}^{-2n\epsilon^2}.$$

4. **Decoding a Bit from a Noisy Signal**

In many wireless communications systems, each receiver listens on a specific frequency. The bit b sent is represented by a $+1$ or -1. Unfortunately, noise from other nearby frequencies can affect the receiver's signal. A simplified model for this noise is as follows. There are n other senders. The ith sender is also trying to send a bit B_i that is represented by $+1$ or -1. The receiver obtains the signal S given by

$$S = b + w \sum_{i=1}^{n} B_i,$$

where w is constant indicating the power of the bits of the other senders.

In order to decode b from S, we use the following scheme: if S is closer to $+1$ than -1, the receiver assumes that the bit sent was a $+1$; if S is closer to -1 than $+1$, the receiver assumes that the bit sent was a -1; if S is equidistant to $+1$ and -1, the receiver fails to recover b.

Assume that all the bits B_i are independent and uniformly distributed over $\{+1, -1\}$.

(a) Show that the probability that the receiver cannot determine b correctly, is at most $2\exp(-\frac{1}{2nw^2})$.
Hint: Transform appropriately each B_i in order to use Problem 3.

(b) If we want to ensure that the probability to correctly determine b is at least $1 - \delta = 0.999$, what condition do we need to impose on the power of the noise w?

(c) What would be the condition on the power of the noise w, if we have used Chebyshev's inequality in order to upper bound the error probability?

(d) Discuss how the analysis of the error probability in (a) compares with the analysis of the error probability using Chebyshev's inequality.

5. **[Bonus] Gaussian Tail Bounds**

The bonus question is just for fun. You are not required to submit the bonus question, but do give it a try and write down your progress.

Let $\phi(y) = \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$ be the PDF of a standard normal random variable $Y \sim \mathcal{N}(0,1)$.

(a) Show that for $y \neq 0$ we have that

$$\phi(y) = -\frac{1}{y} \cdot \phi'(y).$$

(b) Use (a) to show that

$$\mathbb{P}(Y \geq t) \leq \frac{1}{t} \cdot \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}, \quad \text{for all } t > 0.$$

(c) Use part (a) to show that

$$\left(\frac{1}{t} - \frac{1}{t^3}\right) \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} \leq \mathbb{P}(Y \geq t), \quad \text{for all } t > 0.$$