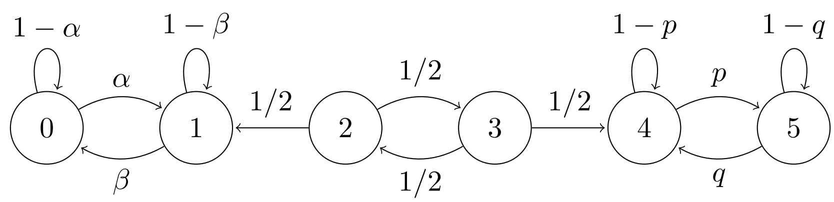
UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 8
Fall 2019

1. Reducible Markov Chain

Consider the following Markov chain, for $\alpha, \beta, p, q \in (0, 1)$.



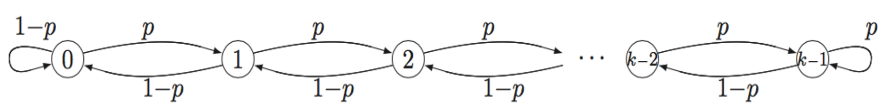
(a) Find all the recurrent and transient classes.

(b) Given that we start in state 2, what is the probability that we will reach state 0 before state 5?

(c) What are all of the possible stationary distributions of this chain?

(d) Suppose we start in the initial distribution $\pi_0 := \begin{bmatrix} 0 & 0 & \gamma & 1-\gamma & 0 & 0 \end{bmatrix}$ for some $\gamma \in [0, 1]$. Does the distribution of the chain converge, and if so, to what?

2. Finite Random Walk

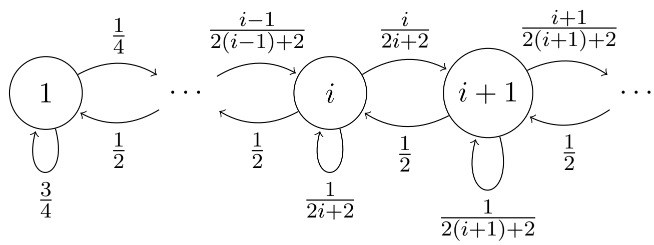


(a) Assume $0 < p < 1$. Find the stationary distribution. *Hint:* Let $q = 1 - p$ and $\rho = \frac{p}{q}$, but be careful when $\rho = 1$.

(b) Find the limit of π_0 and π_{k-1}, the stationary distribution probability for state 0 and $k - 1$, as $k \to \infty$.

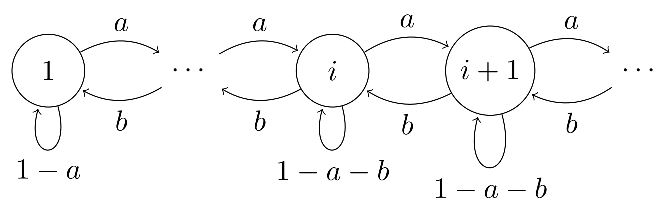
3. Markov Chains with Countably Infinite State Space

(a) Consider a Markov chain with state space $\mathbb{Z}_{>0}$ and transition probability graph:



Show that this Markov chain has no stationary distribution.

(b) Consider a Markov chain with state space $\mathbb{Z}_{>0}$ and transition probability graph:



Assume that $0 < a < b$ and $0 < a + b \leq 1$. Show that the probability distribution given by

$$\pi(i) = \left(\frac{a}{b}\right)^{i-1}\left(1 - \frac{a}{b}\right), \text{ for } i \in \mathbb{Z}_{>0},$$

is a stationary distribution of this Markov chain.

4. **Product of Rolls of a Die**

A fair die with labels (1 to 6) is rolled until the product of the last two rolls is 12. What is the expected number of rolls?

[Hint: You can model this process as a Markov chain with 3 states. Choose your states according to the outcome of last roll. For example, assign one state if it is outcome was 1 or 5 (which is useless if you want the product to be 12). If the outcome was 2,3,4 or 6, it's useful and can be assigned another state. Assign third state to the case when the product last two outcomes was 12.]

5. **Choosing Two Good Movies**

You have a database of a countably infinite number of movies. Each movie has a rating that is uniformly distributed in $\{0, 1, 2, 3, 4, 5\}$ and you want to find two movies such that the sum of their rating is greater than 7.5. Assume that you choose movies from the database one by one and keep the movie with the highest rating so far. You stop when you find that the sum of the ratings of the last movie you have chosen and the movie with the highest rating among all the previous movies is greater than 7.5.

(a) Define an appropriate Markov chain and use the first step equations in order to find the expected number of movies you will have to choose.

(b) Now assume that the ratings of the movies are uniformly distributed in the interval $[0, 5]$. Write the first step equations for the expected number of movies you will have to choose in this case.

6. [Bonus] Choosing Two Good Movies (cont.)

The bonus question is just for fun. You are not required to submit the bonus question, but do give it a try and write down your progress.

Solve the first step equations that you derived in Part (b), in order to find the expected number of movies that you will have to choose.