UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 9
Fall 2019

1. **Metropolis-Hastings**

This problem proves properties of the **Metropolis-Hastings Algorithm**, which you saw in lab.

Recall that the goal of MH was to draw samples from a distribution $p(x)$. The algorithm assumes we can compute $p(x)$ up to a normalizing constant via $f(x)$, and that we have a proposal distribution $g(x, \cdot)$. The steps are:

- Propose the next state y according to the distribution $g(x, \cdot)$.
- Accept the proposal with probability

$$A(x, y) = \min\left\{1, \frac{f(y)}{f(x)} \frac{g(y, x)}{g(x, y)}\right\}.$$

- If the proposal is accepted, then move the chain to y; otherwise, stay at x.

(a) The key to showing why Metropolis-Hastings works is to look at the **detailed balance equations**. Suppose we have a finite irreducible Markov chain on a state space $\mathcal{X}$ with transition matrix P. Show that if there exists a distribution π on $\mathcal{X}$ such that for all $x, y \in \mathcal{X}$,

$$\pi(x)P(x, y) = \pi(y)P(y, x),$$

then π is a stationary distribution of the chain (i.e. $\pi P = \pi$).

(b) Now return to the Metropolis-Hastings chain. What is $P(x, y)$ in this case? For simplicity, assume $x \neq y$.

(c) Show $p(x)$, our target distribution, satisfies the detailed balance equations with $P(x, y)$, and therefore is the stationary distribution of the chain.

(d) If the chain is aperiodic, then the chain will converge to the stationary distribution. If the chain is not aperiodic, we can force it to be aperiodic by considering the **lazy chain**: on each transition, the chain decides not to move with probability 1/2 (independently of the propose-accept step). Explain why the lazy chain is aperiodic, and explain why the stationary distribution is the same as before.

2. The Cut Property and Reversible Markov Chains

(a) For an irreducible Markov chain at stationarity, show that the flow-in equals flow-out relationship holds for any cut of the Markov chain. A cut of a Markov chain is a partition of the states into two disjoint subsets. *Hint*: To solve this problem, try induction on the size of one of the subsets of the cut and write out the flow equations at each step.

(b) The state diagram of Markov Chain is a directed graph, but can be converted to an undirected graph in the following manner.

- For every directed edge in the state diagram, use an undirected edge in the undirected graph.
- If A has a directed edge to B and vice versa, there should still only be one undirected edge.

A sufficient condition for the detailed balance equations to hold is that the resulting graph is a tree. Explain, in words, why this is true.

3. Arrival Times of a Poisson Process

Consider a Poisson process $(N_t, t \geq 0)$ with rate $\lambda = 1$. For $i \in \mathbb{Z}_{>0}$, let S_i be a random variable which is equal to the time of the i-th arrival.

(a) Find $\mathbb{E}[S_3 \mid N_1 = 2]$.

(b) Given $S_3 = s$, where $s > 0$, find the joint distribution of S_1 and S_2.

(c) Find $\mathbb{E}[S_2 \mid S_3 = s]$.

4. Customers in a Store

Consider two independent Poisson processes with rates λ_1 and λ_2. Those processes measure the number of customers arriving in store 1 and 2.

(a) What is the probability that a customer arrives in store 1 before any arrives in store 2?

(b) What is the probability that in the first hour exactly 6 customers arrive, in total, at the two stores?

(c) Given that exactly 6 have arrived, in total, at the two stores, what is the probability that exactly 4 went to store 1?

5. Basketball II

Team A and Team B are playing an untimed basketball game in which the two teams score points according to independent Poisson processes. Team A scores points according to a Poisson process with rate λ_A and Team B scores points according to a Poisson process with rate λ_B. The game is over when one of the teams has scored k more points than the other team. Find the probability that Team A wins.