UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 11
Fall 2020

1. **Exponential MLE, MAP, Hypothesis Testing** Suppose we observe Y distributed according to Exponential(X) where X is either 1 or a. Solve the following hypothesis testing problem:

   Maximize $\mathbb{P}(\hat{X} = 1 \mid X = 1)$ subject to $\mathbb{P}(\hat{X} = 1 \mid X = a) \leq 5\%$

   where $a > 1$ is given.

2. **Basic Properties of Jointly Gaussian Random Variables**

   We explore the relationship between multivariate normal (jointly gaussian) random variables and their characteristic functions. The characteristic function for a multivariate distribution is defined as $\varphi_X(t) = \mathbb{E}[\exp(i\langle t, X\rangle)]$.

   (a) Let Z be a standard normal RV on n variables (that are independent). Show that
   $$\varphi_Z(t) = \exp(-\frac{1}{2}t^\mathsf{T} t).$$

   (b) Recall that any jointly gaussian random variable X can be written as $X = \mu + AZ$, for some mean vector μ and matrix A such that $C = AA^\mathsf{T}$ is the covariance matrix of X. Show that
   $$\varphi_X(t) = \exp(i\langle t, \mu\rangle - \frac{1}{2}t^\mathsf{T} Ct).$$

   (c) Prove that a collection of jointly Gaussian random variables $X_1, \ldots, X_n$ (n is a positive integer) are independent if and only if they are uncorrelated. [*Hint*: use the characteristic function definition, and look at the covariance matrix for uncorrelated RVs.]

   (d) Show that any linear combination of these random variables will be a Gaussian random variable.

3. **Joint Gaussian Probability**

   Let $X \sim \mathcal{N}(1, 1)$ and $Y \sim \mathcal{N}(0, 1)$ be jointly Gaussian with covariance ρ. What is $\mathbb{P}(X > Y)$? [*Hint:* Let $\tilde{X} = X - 1$. Argue that we can write Y as $Y = \rho\tilde{X} + \sqrt{1-\rho^2}Z$ where $Z \sim \mathcal{N}(0, 1)$ is independent, and proceed from there.]