UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 13
Fall 2020

1. Balls in Bins Estimation

We throw $n \geq 1$ balls into $m \geq 2$ bins. Let X and Y represent the number of balls that land in bin 1 and 2 respectively.

(a) Calculate $\mathbb{E}[Y \mid X]$.

(b) What are $L[Y \mid X]$ and $Q[Y \mid X]$ (where $Q[Y \mid X]$ is the best quadratic estimator of Y given X)?

Hint: Your justification should be no more than two or three sentences, no calculations necessary! Think carefully about the meaning of the MMSE.

(c) Unfortunately, your friend is not convinced by your answer to the previous part. Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

(d) Compute $\text{var}(X)$.

(e) Compute $\text{cov}(X, Y)$.

(f) Compute $L[Y \mid X]$ using the formula. Ensure that your answer is the same as your answer to part (b).

2. Gaussian Estimation

Let $Y = X + Z$ and $U = X - Z$, where X and Z are i.i.d. $\mathcal{N}(0, 1)$.

(a) Find the joint distribution of U and Y.

(b) Find the MMSE of X given the observation Y, call this $\hat{X}(Y)$.

(c) Let the estimation error $E = X - \hat{X}(Y)$. Find the conditional distribution of E given Y.

3. Forrest Gump

Forrest Gump is running across the United States, and we would like to track his progress. Assume that on day $n \in \mathbb{N}$ he runs $X(n)$ miles, and the amount he runs each day is determined by the amount he ran on the previous day with some random noise in the following manner: $X(n) = \alpha X(n-1) + V(n)$. Unfortunately, the measurements of the distance he traveled on each day are also subject to some noise. Assume that $Y(n)$ gives the measured number of miles Forrest Gump traveled on day n and that $Y(n) = \beta X(n) + W(n)$. For this problem, assume that $X(0) \sim \mathcal{N}(0, \sigma_X^2), W(n) \sim \mathcal{N}(0, \sigma_W^2), V(n) \sim \mathcal{N}(0, \sigma_V^2)$ are independent.

(a) Suppose that you observe $Y(0)$. Find the MMSE of $X(0)$ given this observation.

(b) Express both $\mathbb{E}[Y(n) \mid Y(0), \dots, Y(n-1)]$ and $\mathbb{E}[X(n) \mid Y(0), \dots, Y(n-1)]$ in terms of $\hat{X}(n-1)$, where $\hat{X}(n-1)$ is the MMSE of $X(n-1)$ given the observations $Y(0), Y(1), \dots, Y(n-1)$.

(c) Show that:

$$\hat{X}(n) = \alpha\hat{X}(n-1) + k_n[Y(n) - \alpha\beta\hat{X}(n-1)]$$

where

$$k_n = \frac{\operatorname{cov}(X(n), \tilde{Y}(n))}{\operatorname{var} \tilde{Y}(n)}$$

and $\tilde{Y}(n) = Y(n) - L[Y(n) \mid Y(0), Y(1), \ldots, Y(n-1)]$.