UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 14
Fall 2020

1. Jump Around

Figure ?? shows the life of Sinho. Some days he is Tired and some days he is Energetic. But he doesn't tell you whether he's Tired or not, and all you can observe is whether he Jumps, Eats, Runs, or Sleeps. We start on day 1 in the Energetic state and there is one transition per day.

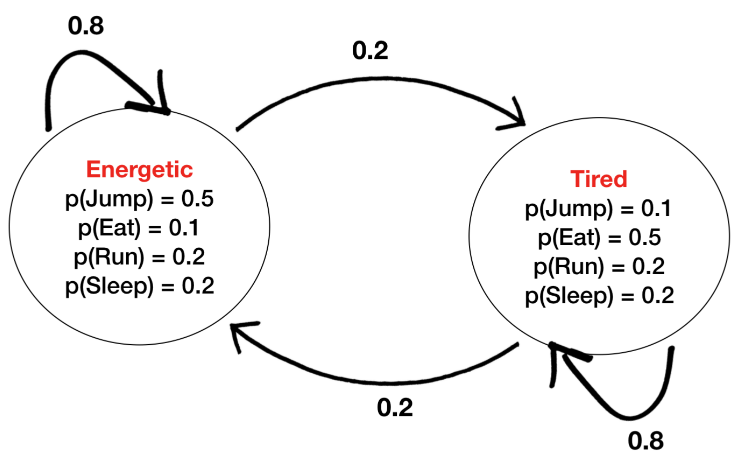

Figure 1: HMM model for Sinho.

For the questions below, we use the following notations:

- q_t: state on day t
- O_t: observation on day t

(a) What is $\mathbb{P}(q_2 = \text{Energetic} \mid O_2 = \text{Eat})$?

(b) What is $\mathbb{P}(O_3 = \text{Sleep} \mid O_2 = \text{Eat})$?

2. Hidden Markov Models

A hidden Markov model (HMM) is a Markov chain $\{X_n\}_{n=0}^{\infty}$ in which the states are considered "hidden" or "latent". In other words, we do not directly observe $\{X_n\}_{n=0}^{\infty}$. Instead, we observe $\{Y_n\}_{n=0}^{\infty}$, where $Q(x, y)$ is the probability that state x will emit observation y. π_0 is the initial distribution for the Markov chain, and P is the transition matrix.

(a) What is $\mathbb{P}(X_0 = x_0, Y_0 = y_0, \dots, X_n = x_n, Y_n = y_n)$, where n is a positive integer, $x_0, \dots, x_n$ are hidden states, and $y_0, \dots, y_n$ are observations?

(b) What is $\mathbb{P}(X_0 = x_0 \mid Y_0 = y_0)$?

(c) We observe $(y_0, \ldots, y_n)$ and we would like to find the most likely sequence of hidden states $(x_0, \ldots, x_n)$ which gave rise to the observations. Let

$$U(x_m, m) = \max_{x_{m+1},\ldots,x_n \in \mathcal{X}} \mathbb{P}(X_m = x_m, X_{m+1:n} = x_{m+1:n}, Y_{0:n} = y_{0:n})$$

denote the largest probability for a sequence of hidden states beginning at state x_m at time $m \in \mathbb{N}$, along with the observations $(y_0, \ldots, y_n)$. Develop a recursion for $U(x_m, m)$ in terms of $U(x_{m+1}, m+1)$, $x_{m+1} \in \mathcal{X}$.

3. Dynamic Programming

A decision problem is characterized by (state, action, noise) (x_k, u_k, w_k) for each positive integer k. The dynamics is governed by: $x_{k+1} = f_k(x_k, u_k, w_k)$, where f_k is some bounded function. Consider N (a positive integer) to be the horizon and the controller wants to minimize a cost function $g_k(x_k, u_k, w_k)$, which is additive over the discrete time step. The terminal cost $g_N(x_N)$ is given. Formulate the problem as a dynamic program and find the total expected cost. Define a policy sequence $\mu = (\mu_0, \ldots, \mu_{N-1})$ as a mapping from state space to action space, i.e., $u_k = \mu_k(x_k)$. Find optimal policy as a function of total expected cost. Also state the dynamic programming update equations for the k-th iteration, using the principle of optimality.