UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 3
Fall 2020

1. **Poisson Merging**

The Poisson distribution is used to model *rare events*, such as the number of customers who enter a store in the next hour. The theoretical justification for this modeling assumption is that the limit of the binomial distribution, as the number of trials n goes to ∞ and the probability of success per trial p goes to 0, such that $np \to \lambda > 0$, is the Poisson distribution with mean λ.

Now, suppose we have two independent streams of rare events (for instance, the number of female customers and male customers entering a store), and we do not care to distinguish between the two types of rare events. Can the combined stream of events be modeled as a Poisson distribution?

Mathematically, let X and Y be independent Poisson random variables with means λ and μ respectively. Prove that $X + Y \sim \text{Poisson}(\lambda + \mu)$. (This is known as **Poisson merging**.) Note that it is **not** sufficient to use linearity of expectation to say that $X + Y$ has mean $\lambda + \mu$. You are asked to prove that the *distribution* of $X + Y$ is Poisson.

Note: You may need to use the binomial theorem which says $(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$. This property will be extensively used when we discuss Poisson processes.

2. **Triangle Density**

Consider random variables X and Y which have a joint PDF uniform on the triangle with vertices at $(0, 0), (1, 0), (0, 1)$.

(a) Find the joint PDF of X and Y.
(b) Find the marginal PDF of Y.
(c) Find the conditional PDF of X given Y.
(d) Find $\mathbb{E}[X]$ in terms of $\mathbb{E}[Y]$.
(e) Find $\mathbb{E}[X]$.

3. **Change of Variables**

Let X be a R.V with PDF $f_X(x)$ and CDF $F_X(x)$. Let $g(X)$ be an invertible function. The change of variables problem asks for the density of $Y = g(X)$ which is a new R.V. To find this distribution, we use the definition of the CDF

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$$

How would you do this if the function g was not invertible? For example $g(x) = x^2 \forall x \in \mathbb{R}$ has two values in the domain mapping to each value in its range. In these cases we have to include all parts of the domain that contribute to the probability. In the case of a discrete distribution using the above g, this would look like

$$P(Y = y) = P(g(X) = y) = P(X \in \{-\sqrt{y}, \sqrt{y}\})$$

(a) Suppose that X has the **standard normal distribution**, that is, X is a continuous random variable with density function

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{x^2}{2}\Big).$$

What is the density function of $\exp X$? (The answer is called the **lognormal distribution**.)

(b) Suppose that X is a continuous random variable with density f. What is the density of X^2?

(c) What is the answer to the previous question when X has the standard normal distribution? (This is known as the **chi-squared distribution**.)