EECS 126 Probability and Random Processes
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Midterm 1

Last Name	First Name	SID

Rules.

- **Unless otherwise stated, all your answers need to be justified and your work must be shown. Answers without sufficient justification will get no credit.**
- You have 70 minutes to complete the exam and 10 minutes exclusively for submitting your exam to Gradescope. (DSP students with $X\%$ time accomodation should spend $70 \cdot X\%$ time on the exam and 10 minutes to submit).
- Collaboration with others is strictly prohibited.
- You may reference your notes, the textbook, and any material that can be found through the course website. You may use Google to search up general knowledge. However, searching up a question is not allowed.
- You may not use online solvers or graphing tools (ex. WolframAlpha, Desmos, Python). Simple functions (ex. combinations, multiplication) are fine.
- For any clarifications you have, please create a private Piazza post. We will have a Google Doc that shows our official clarifications.

Problem	points earned	out of
Honor Code		1
Problem 1		60
Problem 2		27
Problem 3		14
Total		102

1 Assorted Problems

(a) **Lights**

Suppose there are n lights that turn on at the same time, each of which have a lifespan modeled by an exponential distribution with parameter λ. Let $t_1, t_2, \ldots, t_{n-1}$ describe the amount of time elapsed between successive failures, where t_i is the time between the i-th failure and the $i+1$-th failure. Find the expected value of the minimum of all the t_i $(1 \leq i \leq n-1)$.

Example: Say we have 4 lights that shutoff at times 2, 4, 7, and 8. The times elapsed between shutoffs are 2, 3, and 1.

The minimum of n i.i.d. exponential random variables is an exponential parameterized by the sum of the individual rates. By the memoryless property, each t_i is distributed as an exponential with parameter $(n-i+1)\lambda$ as it sums the remaining number of lights left to fail. Hence the minimum has rate $(1+2+...+n-1)\lambda$, which has expected value $\frac{2}{n(n-1)\lambda}$.

(b) **Geometric**

Suppose that on any given day there is a wildfire with probability p, and all the days are independent. Given that there is at least one wildfire within the next n days, what is the expected value of the number of days until the next wildfire (i.e. a value between 1 and n possibly dependent on p and/or n)?

We want to calculate $\mathrm{E}[X|X \leq n]$ for $X \sim \text{Geometric}(p)$. Notice that we can express

$$\mathrm{E}[X] = \mathrm{E}[X|X \leq n] \cdot \mathrm{P}(X \leq n) + \mathrm{E}[X|X > n] \cdot \mathrm{P}(X > n)$$

We know that $\mathrm{E}[X] = \frac{1}{p}$, and since the geometric distribution is memoryless, $\mathrm{E}[X|X > n] = n + \frac{1}{p}$. Moreover, $\mathrm{P}(X > n) = (1-p)^n$, and $\mathrm{P}(X \leq n) = 1 - (1-p)^n$. Thus, we conclude that

$$\mathrm{E}[X|X \leq n] = \frac{\mathrm{E}[X] - \mathrm{E}[X|X > n] \cdot \mathrm{P}(X > n)}{\mathrm{P}(X \leq n)} = \boxed{\frac{\frac{1}{p} - \left(n + \frac{1}{p}\right)(1-p)^n}{1-(1-p)^n}}$$

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(c) **b-Reasonable**

Let's say that for some real $b \geq 1$ that a random variable X is b-reasonable if it satisfies:

$$\mathbb{E}[X^4] \leq b(\mathbb{E}[X^2])^2$$

Suppose X is b-reasonable and that $\mathbb{E}[X] = 0$ and $\mathrm{var}(X) = \sigma^2$. Prove that

$$\Pr[|X| \geq t\sigma] \leq \frac{b}{t^4}$$

Applying Markov's inequality to X^4:

$$\Pr[|X| \geq t\sigma] = \Pr[|X| \geq t\sqrt{\mathbb{E}[X^2]}] = \Pr[X^4 \geq t^4\mathbb{E}[X^2]^2] \leq \frac{\mathbb{E}[X^4]}{t^4\mathbb{E}[X^2]^2} \leq \frac{b}{t^4}$$

Note how this is better than the normal Chebyshev bound by a factor of $O(\frac{1}{t^2})$.

(d) **Normal Product Sequence**

Let X_1 have some distribution whose moments (i.e. $\mathrm{E}[X_1], \mathrm{E}[X_1^2], \mathrm{E}[X_1^3], ...$) are all known. For $i \geq 2$, $X_i \sim \mathcal{N}(X_1 X_2 ... X_{i-1}, 1)$, i.e. a normal distribution with mean equal to the product of the previous elements of the sequence. Find the expectation of X_4 in terms of moments of X_1.

Recall that the second moment of a $\mathcal{N}(\mu, 1)$ RV is $\mu^2 + 1$, which we'll use below to calculate $\mathrm{E}[X_2^2|X_1]$.

Applying the law of iterated expectation:

$$\begin{aligned}
\mathbb{E}[X_1X_2X_3] &= \mathbb{E}[\mathbb{E}[X_1X_2X_3|X_1X_2]] \\
&= \mathbb{E}[X_1X_2\mathbb{E}[X_3|X_1X_2]] \\
&= \mathbb{E}[X_1^2X_2^2] \\
&= \mathbb{E}[\mathbb{E}[X_1^2X_2^2|X_1]] \\
&= \mathbb{E}[X_1^2\mathbb{E}[X_2^2|X_1]] \\
&= \mathbb{E}[X_1^2(X_1^2+1)] \\
&= \mathbb{E}[X_1^2] + \mathbb{E}[X_1^4]
\end{aligned}$$

(e) **Pascal's Coins**

Someone tells you that they flipped $n \geq 1$ fair coins and got 4 heads, but they don't tell you what n is. What is the smallest integer MLE estimate of n?

Note that the pmf of the binomial distribution given n is $\mathbb{P}(h|n) = \frac{n!}{(n-h)!h!} \cdot \frac{1}{2^n}$.

As we increase $n \to n+1$, the left side grows by $\frac{n+1}{n-3}$ but the right side shrinks by $\frac{1}{2}$, which means that this quickly converges to zero as n gets larger. Hence we just have to check the first couple values of n. The smallest MLE estimate is 7.

(f) **Convergence**

Let Z_i equal i^4 with probability $\frac{1}{i^2}$ and -1 with probability $1 - \frac{1}{i^2}$, and let $W_n = \sum_{i=1}^n Z_i$. Show that the probability of $W_n \to -\infty$ as $n \to \infty$ is one. (Hint: $\sum_{i=1}^\infty \frac{1}{i^2} = \frac{\pi^2}{6}$)

Let our wealth be denoted as $W_n = \sum_{i=1}^n Z_i$, where Z_i represents our profits from day i. We have $\mathbb{P}(S_n \to -\infty)$ is the same as $\mathbb{P}(\{Z_i = -1\ i.o.\}) = 1$ and $\mathbb{P}(\{Z_i > -1\ i.o.\} = 0$. Then we calculate $\sum_{i=1}^\infty \mathbb{P}(Z_i > -1) = \frac{\pi^2}{6} < \infty$. By the Borel-Cantelli lemma, $S_n \to -\infty$ with probability one, so our wealth converges to $-\infty$.

Note how this is different than the expected value, which goes to positive infinity, which is an example of how the expected value can be very misleading.

The alternate version of this problem with $Z_i = (i!)^2$ with probability $\frac{1}{i!}$ follows the same line of logic, as the sum of the probabilities is still finite.

2 Chess Antics

(a) **Knight's Journey**

Consider a 13×13 chess board. An "upward knight", starting in the bottom left corner $(1, 1)$, tries to move to the upper right corner $(13, 13)$. This knight can only make one of two moves:

(a) It can move up by two, right by one: $(x, y) \to (x + 1, y + 2)$

(b) It can move up by one, right by two: $(x, y) \to (x + 2, y + 1)$

If both moves are available (would not take the knight out of bounds), it chooses one uniformly at random, or else it takes the only available move. What is the probability the knight travels through the center square $(7, 7)$? (**Hint**: think how many of each type of move is required to reach $(13, 13)$ or $(7, 7)$)

Let's denote the first type of move as U, and the second type as R. Note that the knight's position does not depend on the ordering of its prior moves, and is only determined by the number of U moves and R moves. A full journey consists of 4 of each move; reaching the center square consists of 2 of each move. Hence we can count the number of paths that travel through the center as $(\binom{4}{2})^2 = 36$. The total number of paths is $\binom{8}{4} = \frac{8*7*6*5}{4*3*2*1} = 70$. The probability is therefore $\frac{18}{35}$.

(b) **Queen's Journey**

Suppose we have a queen on a 3×3 chess board. The queen's initial position is random, and it proceeds to make random legal moves (the queen can move horizontally, vertically, or diagonally for any number of squares). Assume the queen never stays in the same square. What is the fraction of time the queen spends in any of the four corners of the board as its number of moves approaches infinity? (**Hint**: can this be modeled as a Markov chain?)

By the Big Theorem for finite state MCs, since the MC is irreducible, this problem reduces to simply solving for the invariant/stationary distribution.

Solution 1. Random walk on undirected graph.

This is just a random walk on a graph, so the stationary distribution is just for a particular square on the board is

$$\pi_{i,j} = \frac{\text{number of legal moves from square } (i, j)}{\text{sum of legal moves from every square}}$$

It remains to note that there are 6 legal moves from any square on the border of the board, while 8 legal moves from the center of the board. Therefore our answer is

$$\frac{6 \cdot 4}{6 \cdot 8 + 8} = \frac{24}{56} = \frac{3}{7}$$

Solution 2. Solving the invariant distribution directly.

Notice that, by symmetry, we can group the 9 states into three groups: the corner squares, the center (middle) square, and the center squares of the edges; let's denote these by C, M, and E, respectively. Now we just have to solve for the invariant distribution of this 3-state Markov chain. The transition probabilities follow:

$$\begin{aligned} \pi(C) &= \frac{1}{2}\pi(C) + \frac{1}{3}\pi(E) + \frac{1}{2}\pi(M) \\ \pi(E) &= \frac{1}{3}\pi(C) + \frac{1}{2}\pi(E) + \frac{1}{2}\pi(M) \\ \pi(M) &= \frac{1}{6}\pi(C) + \frac{1}{6}\pi(E) \\ 1 &= \pi(C) + \pi(E) + \pi(M) \end{aligned}$$

Substituting the third equation into the fourth equation gives $\pi(M) = \frac{1}{7}$ for the center square. Then, by symmetry or plugging into the remaining equations, $\pi(C) = \pi(E) = \frac{3}{7}$.

(c) **Final Showdown**

Kevin and Michael are playing a chess match consisting of many games. Let K equal the number of points Kevin has won minus the number of points Michael has won. Both players are equally matched, so Kevin and Michael are both equally likely to win each game. If Kevin wins a game, Kevin gets a point; if Michael wins, Michael gets a point; there are no draws. The match ends when $K = 5$ (Kevin wins) or when $K = -5$ (Michael wins).

(a) Let K_t denote the value of K after t games have been played and let $K_{t+1} = K_t$ if the match has already ended by time t. Draw K_t as a Markov chain and find the corresponding transition matrix P (such that $\pi_{t+1} = P\pi_t$ models the distribution of K_{t+1}).

For $-4 \leq i \leq 4$: $P_{i,i} = 0, P_{i,i+1} = \frac{1}{2}, P_{i,i-1} = \frac{1}{2}$
For $i = -5, 5$: $P_{i,i} = 1$

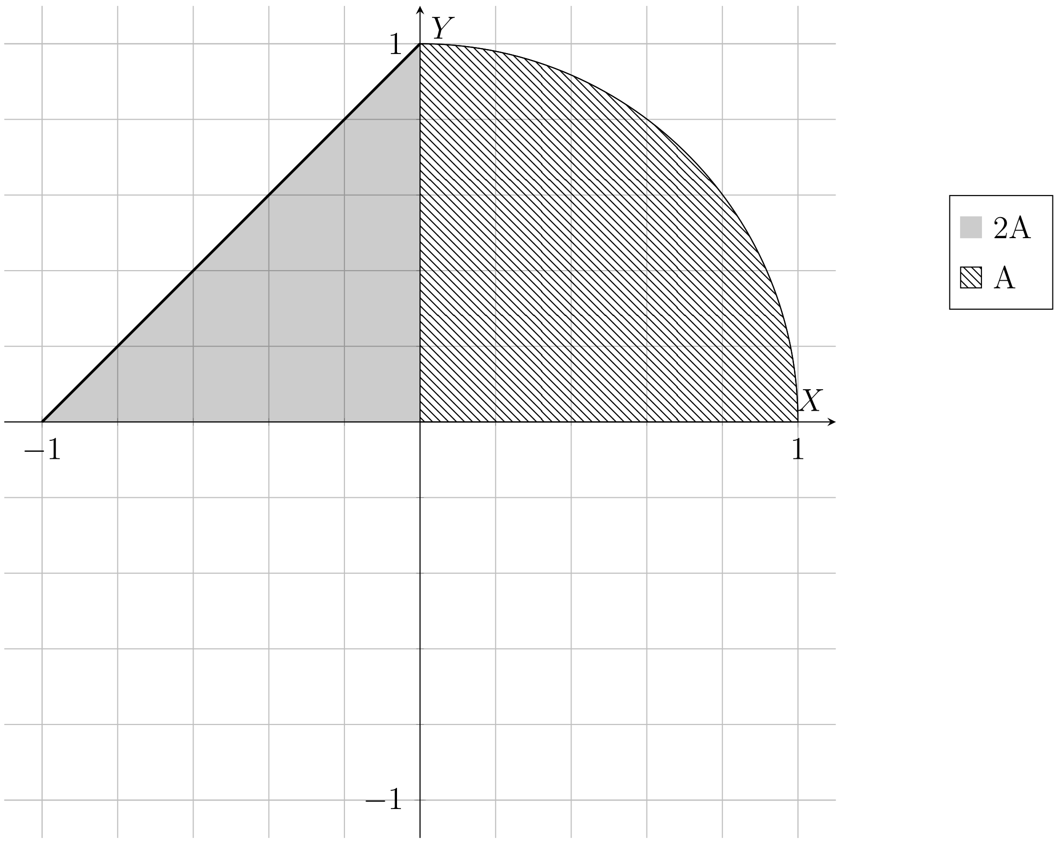
(b) Is the Markov chain irreducible? What are the states that have can nonzero probability in the stationary/invariant distribution?

No; the two end states -5 and 5 are sink states, forming their own two recurrent classes, while the middle states belong to their own class. Hence the MC is not irreducible.

The only two states with nonzero stationary probability are -5 and 5, because they have self-loops with probability 1 and the other states feed into them.

3 Graphical Density A

Two random variables X and Y have a joint density as pictured below:



Note that the formula for a circle of radius 1 centered at the origin is $x^2 + y^2 = 1$, and the right portion of the graph is just one quarter of a unit circle.

(a) What is A? [2 points]

(b) Find the marginals $f_X(x)$ and $f_Y(y)$. Write your answer in terms of A. [4 points]

(c) What is $f_{Y|X}(y|x)$? [2 points]

(d) What is $\mathbb{E}[Y|X]$? [2 points]

(e) What is $\mathbb{E}[Y]$? Write your final answer in terms of A. [4 points]

1. We have that the total volume must integrate to 1, so $2A * 1/2 + A\pi/4 = 1 \Rightarrow A = \frac{1}{1+\pi/4}$
2. We have that

$$f_Y(y) = \int_{y-1}^{0} 2A dx + \int_{0}^{\sqrt{1-y^2}} A dx$$
$$= 2(1-y)A + A\sqrt{1-y^2}$$

And for $x \in (-1, 0)$:

$$f_X(x) = \int_0^{x+1} 2A dy = 2A(x+1)$$

And for $x \in (0, 1)$:

$$f_X(x) = \int_0^{\sqrt{1-x^2}} A dy = A\sqrt{1-x^2}$$

So in total we have

$$f_X(x) = \begin{cases} 2A(x+1) & x \in (-1,0) \\ A\sqrt{1-x^2} & x \in (0,1) \end{cases}$$

3. $f_{Y|X}(y|x)$ is simply uniform over the line that is valid. So

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x+1} & x \in (-1,0) \\ \frac{1}{\sqrt{1-x^2}} & x \in (0,1) \end{cases}$$

4.

$$\mathbb{E}[Y|X] = \begin{cases} \frac{x+1}{2} & x \in (-1,0) \\ \frac{\sqrt{1-x^2}}{2} & x \in (0,1) \end{cases}$$

5. We have

$$\mathbb{E}[Y] = \int f_X(x)\mathbb{E}[Y|X=x]dx$$

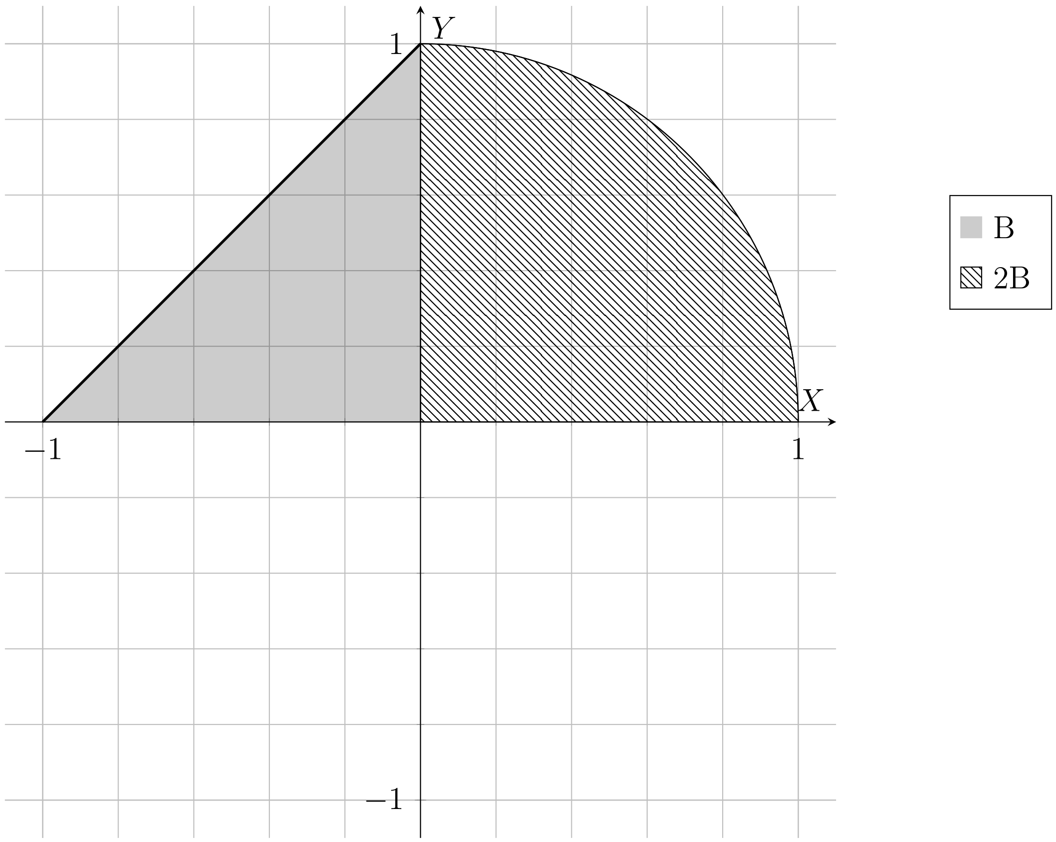
$$= \int_{-1}^{0} \frac{x+1}{2} 2A(x+1)dx + \int_0^1 \frac{\sqrt{1-x^2}}{2} A\sqrt{1-x^2}$$

$$= A(1/3x^3 + x^2 + x)\Big|_{-1}^{0} + \frac{A}{2}(x - x^3/3)\Big|_0^1$$

$$= \frac{2A}{3}$$

4 Graphical Density B

Two random variables X and Y have a joint density as pictured below:



Note that the formula for a circle of radius 1 centered at the origin is $x^2 + y^2 = 1$, and the right portion of the graph is just one quarter of a unit circle.

(a) What is B? [2 points]

(b) Find the marginals $f_X(x)$ and $f_Y(y)$. Write your answers in terms of B. [4 points]

(c) What is $f_{Y|X}(y|x)$? [2 points]

(d) What is $\mathbb{E}[Y|X]$? [2 points]

(e) What is $\mathbb{E}[Y]$? Write your final answer in terms of B. [4 points]

1. We have that the total volume must integrate to 1, so $B * 1/2 + 2B\pi/4 = 1 \Rightarrow B = \frac{2}{1+\pi}$
2. We have that

$$f_Y(y) = \int_{y-1}^{0} B dx + \int_{0}^{\sqrt{1-y^2}} 2B dx$$
$$= (1-y)B + 2B\sqrt{1-y^2}$$

And for $x \in (-1, 0)$:

$$f_X(x) = \int_0^{x+1} B dy = B(x+1)$$

And for $x \in (0, 1)$:

$$f_X(x) = \int_0^{\sqrt{1-x^2}} 2B dy = 2B\sqrt{1-x^2}$$

So in total we have

$$f_X(x) = \begin{cases} B(x+1) & x \in (-1, 0) \\ 2B\sqrt{1-x^2} & x \in (0, 1) \end{cases}$$

3. $f_{Y|X}(y|x)$ is simply uniform over the line that is valid. So

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x+1} & x \in (-1, 0) \\ \frac{1}{\sqrt{1-x^2}} & x \in (0, 1) \end{cases}$$

4.
$$\mathbb{E}[Y|X] = \begin{cases} \frac{x+1}{2} & x \in (-1, 0) \\ \frac{\sqrt{1-x^2}}{2} & x \in (0, 1) \end{cases}$$

5. We have

$$\mathbb{E}[Y] = \int f_X(x)\mathbb{E}[Y|X = x]dx$$

$$= \int_{-1}^{0} \frac{x+1}{2} B(x+1)dx + \int_0^1 \frac{\sqrt{1-x^2}}{2} 2B\sqrt{1-x^2}$$

$$= B/2(1/3x^3 + x^2 + x)\Big|_{-1}^{0} + B(x - x^3/3)\Big|_0^1$$

$$= \frac{5B}{6}$$