UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 13
Fall 2020

1. Gaussian Random Vector MMSE

Let
$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$$
be a Gaussian random vector.

Let
$$W = \begin{cases} 1, & \text{if } Y > 0 \\ 0, & \text{if } Y = 0 \\ -1, & \text{if } Y < 0 \end{cases}$$
be the sign of Y. Find $\mathbb{E}[WX \mid Y]$.

2. Geometric MMSE

Let N be a geometric random variable with parameter $1-p$, and $(X_i)_{i\in\mathbb{N}}$ be i.i.d. exponential random variables with parameter λ. Let $T = X_1 + \cdots + X_N$. Compute the LLSE and MMSE of N given T.

Hint: Compute the MMSE first.

3. Property of MMSE

Let $X, Y_1, \ldots, Y_n$ be square integrable random variables. Argue that
$$\mathbb{E}\big[(X - \mathbb{E}[X \mid Y_1, \ldots, Y_n])^2\big] \leq \mathbb{E}\left[\left(X - \sum_{i=1}^{n} \mathbb{E}[X \mid Y_i]\right)^2\right].$$

4. Stochastic Linear System MMSE

Let $(V_n,\ n \in \mathbb{N})$ be i.i.d. $\mathcal{N}(0, \sigma^2)$ and independent of $X_0 = \mathcal{N}(0, u^2)$. Let $|a| < 1$. Define
$$X_{n+1} = aX_n + V_n, \qquad n \in \mathbb{N}.$$

(a) What is the distribution of X_n, where n is a positive integer?

(b) Find $\mathbb{E}[X_{n+m} \mid X_n]$ for $m, n \in \mathbb{N}$, $m \geq 1$.

(c) Find u so that the distribution of X_n is the same for all $n \in \mathbb{N}$.

5. Error of the Kalman Filter for a Linear Stochastic System

The linear stochastic system
$$\begin{bmatrix} X_{1,k+1} \\ X_{2,k+1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} X_{1,k} \\ X_{2,k} \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} w_k, \qquad k \geq 0,$$

starts from an arbitrary (known) initial condition $\begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix}$ and the system noise variables $(w_k,\ k \geq 0)$ are i.i.d. normal with mean 0 and variance 1.

The state variables are not directly observable. However, we can observe

$$Y_k = X_{1,k} + X_{2,k}, \qquad k \geq 0.$$

Let $\hat{X}_{k|k}$ denote the minimum mean square error estimator of $X_k = \begin{bmatrix} X_{1,k} \\ X_{2,k} \end{bmatrix}$ given $(Y_0, \ldots, Y_k)$. Determine the asymptotic behavior of the covariance matrix of the estimation error.

Note: This problem needs thought. Note that there is no observation noise, so the assumption used in the derivation of the Kalman filter equations, that the covariance matrix of the observation noise is positive definite, is no longer valid.

6. Random Walk with Unknown Drift

Consider a random walk with unknown drift. The dynamics are given, for $n \in \mathbb{N}$, as

$$\begin{aligned} X_1(n+1) &= X_1(n) + X_2(n) + V(n), \\ X_2(n+1) &= X_2(n), \\ Y(n) &= X_1(n) + W(n). \end{aligned}$$

Here, X_1 represents the position of the particle and X_2 represents the velocity of the particle (which is unknown but constant throughout time). Y is the observation. V and W are independent Gaussian noise variables with mean zero and variance σ_V^2 and σ_W^2 respectively.

(a) Write down the dynamics of the system in matrix-vector form and write down the Kalman filter recursive equations for this system.

(b) Let k be a positive integer. Compute the prediction $\mathbb{E}(X(n+k) \mid Y^{(n)})$, where $Y^{(n)}$ is the history of the observations $Y_0, \ldots, Y_n$, in terms of the estimate $\hat{X}(n) := \mathbb{E}(X(n) \mid Y^{(n)})$.

(c) Now let $k = 1$ and compute the smoothing estimate $\mathbb{E}(X(n) \mid Y^{(n+1)})$ in terms of the quantities that appear in the Kalman filter equation.

Hint: Use the law of total expectation

$$\mathbb{E}\big(X(n) \mid Y^{(n+1)}\big) = \mathbb{E}\big[\mathbb{E}\big(X(n) \mid X(n+1), Y^{(n+1)}\big) \mid Y^{(n+1)}\big],$$

as well as the *innovation*

$$\tilde{X}(n+1) := X(n+1) - L[X(n+1) \mid Y^{(n)}].$$