UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 2
Fall 2020

1. Secret Hitler

In the game of Secret Hitler, there are 9 players. 4 of them are Fascist and 5 of them are Liberal. There is also a deck of 17 cards containing 11 Fascist "policies" and 6 Liberal "policies". Fascists want to play Fascist policies, and Liberals want to play Liberal policies. Here's how the play proceeds.

- A President and a Chancellor are chosen uniformly at random from the 9 players.
- The President draws 3 policies from the deck and gives 2 to the Chancellor.
- The Chancellor chooses one to play.

Now suppose you are the Chancellor, but the President gave you 2 Fascists. Being a Liberal, you wonder, did the President just happen to have 3 Fascist policies, or was the President a Fascist who secretly discarded a Liberal policy. In this scenario, what's the probability that the President is Fascist? Let's assume that Fascist presidents always try to discard Liberal policies.

2. Exam Difficulties

The difficulty of an EECS 126 exam, Θ, is uniformly distributed on $[0, 100]$, and Alice gets a score X that is uniformly distributed on $[0, \Theta]$. Alice gets her score back and wants to estimate the difficulty of the exam.

(a) What is the LLSE for Θ?

(b) What is the MAP of Θ?

3. Bounds for the Coupon Collector's Problem

Recall the coupon collector's problem, where X is a random variable which is equal to the number of boxes bought until one of every type of coupon is obtained (there are n total coupons).

The expected value of X is nH_n, where H_n is the *harmonic number of order* n which is defined as

$$H_n \triangleq \sum_{i=1}^{n} \frac{1}{i},$$

and satisfies the inequalities

$$\ln n \leq H_n \leq \ln n + 1.$$

(a) Use Markov's inequality in order to show that

$$\mathbb{P}(X > 2nH_n) \leq \frac{1}{2}.$$

(b) Use Chebyshev's inequality in order to show that

$$\mathbb{P}(X > 2nH_n) \le \frac{\pi^2}{6(\ln n)^2}.$$

Note: You can use the identity

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}.$$

(c) Define appropriate events and use the union bound in order to show that

$$\mathbb{P}(X > 2nH_n) \le \frac{1}{n}.$$

Note: The sequence $a_n = (1 - 1/n)^n$, for $n = 1, 2, 3, \ldots$, is strictly increasing and $\lim_{n\to\infty} a_n = 1/\mathrm{e}$.

4. **Message Segmentation**

The number of bytes N in a message has a geometric distribution with parameter p. Suppose that the message is segmented into packets, with each packet containing m bytes if possible, and any remaining bytes being put in the last packet. Let Q denote the number of full packets in the message, and let R denote the nubmer of bytes left over.

(a) Find the joint PMF of Q and R. Pay attention on the support of the joint PMF.

(b) Find the marginal PMFs of Q and R.

(c) Repeat part (b), given that we know that $N > m$.

Note: you can use the formulas

$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}, \text{ for } a \ne 1$$
$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}, \text{ for } |x| < 1$$

in order to simplify your answer.

5. **Random Bipartite Graph**

Consider a random bipartite graph with, K left nodes and M right nodes. Each of the $K \cdot M$ possible edges of this graph is present with probability p independently.

(a) Find the distribution of the degree of a particular right node.

(b) Now suppose you pick an edge which is present in the graph. What is the distribution of the degree of the right node connected to this edge? Is it the same as in part (a)?

(c) We call a right node with degree one a *singleton*. What is the average number of singletons in a random bipartite graph?

(d) Find the average number of left nodes that are connected to at least one singleton.

6. **Compact Arrays**

Consider an array of n entries, where n is a positive integer. Each entry is chosen uniformly randomly from $\{0, \ldots, 9\}$. We want to make the array more compact, by putting all of the non-zero entries together at the front of the array. As an example, suppose we have the array

$$[6, 4, 0, 0, 5, 3, 0, 5, 1, 3].$$

After making the array compact, it now looks like

$$[6, 4, 5, 3, 5, 1, 3, 0, 0, 0].$$

Let i be a fixed positive integer in $\{1, \ldots, n\}$. Suppose that the ith entry of the array is non-zero (assume that the array is indexed starting from 1). Let X be a random variable which is equal to the index that the ith entry has been moved after making the array compact. Calculate $\mathbb{E}[X]$ and $\text{var}(X)$.

7. **Almost fixed points of a permutation**

Let Ω be the set of all permutations of the numbers $1, 2, ..., n$. Let an almost fixed point be defined as follows: If we put the numbers $i \in 1, 2, ..., n$ around a circle in clockwise order (such that 1 and n are next to each other) and then assign another number $\omega(i) \in 1, 2, ..., n$ to it, if the number $\omega(i)$ is next to i (or is equal to i), we will say that i is almost a fixed point. So, for the permutation $\omega(1) = 5, \omega(2) = 3, \omega(3) = 1, \omega(4) = 4, \omega(5) = 2$, we have that 1, 2, and 4 are almost fixed points.

Now, let $X(\omega)$ denote the number of almost fixed points in $\omega \in \Omega$. Find $\mathbb{E}[X]$ and $\text{var}(X)$. You may assume that $n \geq 5$.