UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Problem Set 4 (optional)
Fall 2020

1. Markov Chain Practice

Consider a Markov chain with three states 0, 1, and 2. The transition probabilities are $P(0,1) = P(0,2) = 1/2$, $P(1,0) = P(1,1) = 1/2$, and $P(2,0) = 2/3$, $P(2,2) = 1/3$.

(b) In the long run, what fraction of time does the chain spend in state 1?

(c) Suppose that X_0 is chosen according to the steady state distribution. What is $\mathbb{P}(X_0 = 0 \mid X_2 = 2)$?

2. Backwards Markov Property

Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain with state space $\mathcal{S}$. Show that for every $m, k \in \mathbb{N}$, with $m \geq 1$, we have

$$\mathbb{P}(X_k = i_0 \mid X_{k+1} = i_1, \ldots, X_{k+m} = i_m) = \mathbb{P}(X_k = i_0 \mid X_{k+1} = i_1),$$

for all states $i_0, i_1, \ldots, i_m \in \mathcal{S}$.

3. Product of Rolls of a Die

A fair die with labels (1 to 6) is rolled until the product of the last two rolls is 12. What is the expected number of rolls?

[Hint: You can model this process as a Markov chain with 3 states. Choose your states according to the outcome of last roll. For example, assign one state if it is outcome was 1 or 5 (which is useless if you want the product to be 12). If the outcome was 2,3,4 or 6, it's useful and can be assigned another state. Assign third state to the case when the product last two outcomes was 12.]

4. Fly on a Graph

A fly wanders around on a graph G with vertices $V = \{1, \ldots, 5\}$, shown in Figure **??**.

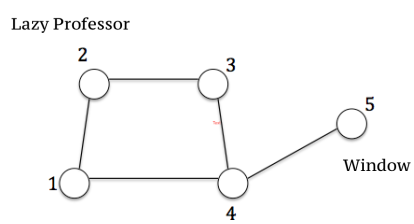

Figure 1: A fly wanders randomly on a graph.

(a) Suppose that the fly wanders as follows: if it is at node i at time n, then it chooses one of its neighbors j of i uniformly at random, and then wanders to node j at time $n+1$. For times $n = 0, 1, 2, \ldots$, let X_n be the fly's position at time n. Argue that $\{X_n,\ n \in \mathbb{N}\}$ is a Markov chain, and find the invariant distribution.

(b) Now for the process in part (a), suppose that the (not-to-be-named) professor sits at node 2 reading a heavy book. The professor is very lazy, so they don't move at all, but will drop the book on the fly if it reaches node 2 (killing it instantly). On the other hand, node 5 is a window that lets the fly escape. What is the probability that the fly escapes through the window supposing that it starts at node 1?

(c) Now suppose that the fly wanders as follows: when it is at node i at time n, it chooses uniformly from all neighbors of node i except for the one that it just came from. For times $n = 0, 1, 2, \ldots$, let Y_n be the fly's position at time n. Is this new process $\{Y_n, n \in \mathbb{N}\}$ a Markov chain? If it is, write down the probability transition matrix; if not, explain why it does not satisfy the definition of Markov chains.

5. Gambling Game

Let's play a game. You stake a positive initial amount of money w_0. You toss a fair coin. If it is heads you earn an amount equal to three times your stake, so you quadruple your wealth. If it comes up tails you lose everything. There is one requirement though, you are not allowed to quit and have to keep playing, by staking all your available wealth, over and over again.

Let W_n be a random variable which is equal to your wealth after n plays.

(a) Find $\mathbb{E}[W_n]$ and show that $\lim_{n\to\infty} \mathbb{E}[W_n] = \infty$.

(b) Since $\lim_{n\to\infty} \mathbb{E}[W_n] = \infty$, this game sounds like a good deal! But wait a moment!! Where does the sequence of random variables $\{W_n\}_{n\geq 0}$ converge to?