UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 5
Fall 2020

1. Midterm

Solve all of the problems on the midterm again (including the ones you got correct).

2. Interesting Bernoulli Convergence

Consider an independent sequence of random variables where $X_n \sim B(\frac{1}{n})$.

(a) Prove that X_n converges to 0 in probability.

(b) Prove that X_n **does not** converge almost surely to 0.

3. More Almost Sure Convergence

(a) Suppose that, with probability 1, the sequence $(X_n)_{n\in\mathbb{N}}$ oscillates between two values $a \neq b$ infinitely often. Is this enough to prove that $(X_n)_{n\in\mathbb{N}}$ does *not* converge almost surely? Justify your answer.

(b) Suppose that Y is uniform on $[-1, 1]$, and X_n has distribution

$$\mathbb{P}\big(X_n = (y + n^{-1})^{-1} \bigm| Y = y\big) = 1.$$

Does $(X_n)_{n=1}^{\infty}$ converge a.s.?

(c) Define random variables $(X_n)_{n\in\mathbb{N}}$ in the following way: first, set each X_n to 0. Then, for each $k \in \mathbb{N}$, pick j uniformly randomly in $\{2^k, \dots, 2^{k+1} - 1\}$ and set $X_j = 2^k$. Does the sequence $(X_n)_{n\in\mathbb{N}}$ converge a.s.?

(d) Does the sequence $(X_n)_{n\in\mathbb{N}}$ from the previous part converge in probability to some X? If so, is it true that $\mathbb{E}[X_n] \to \mathbb{E}[X]$ as $n \to \infty$?

4. Final Showdown

Here we extend upon one of the problems from the midterm; you should use the results from that problem to start this one.

Kevin and Michael are playing a chess match consisting of many games. Let K equal the number of points Kevin has won minus the number of points Michael has won. Both players are equally matched, so Kevin and Michael are both equally likely to win each game. If Kevin wins a game, Kevin gets a point; if Michael wins, Michael gets a point; there are no draws. The match ends when $K = 5$ (Kevin wins) or when $K = -5$ (Michael wins). Let K_0 denote the score of Kevin at the start of the match.

Show that the probability that Kevin wins depends on K_0, i.e. the initial distribution of the Markov chain. In particular, show that the probability that Kevin wins is $\frac{K_0+5}{10}$.

5. **Twitch Plays Pokemon**

You wake up one day and are surprised to see that it is 2014, when the world was peaceful. You immediately start playing Twitch Plays Pokemon. Suddenly, you realized that you may be able to analyze Twitch Plays Pokemon.

You		
		Stairs

Figure 1: Part (a)

(a) The player in the top left corner performs a random walk on the 8 checkered squares and the square containing the stairs. At every step the player is equally likely to move to any of the squares in the four cardinal directions (North, West, East, South) if there is a square in that direction. Find the expected number of moves until the player reaches the stairs in Figure 1.

[Hint: Use symmetry to reduce the number of states in your Markov chain]

You		
Stairs		Stairs

Figure 2: Part (b)

(b) The player randomly walks in the same way as in the previous part. Find the probability that the player reaches the stairs in the bottom right corner in Figure 2.

[Hint: For each squared box, assign a variable that denotes the probability of reaching the "good" stairs. Use symmetry again to reduce the number of such variables.]

Hint: For both problems use symmetry to reduce the number of states and variables. The equations are very reasonable to solve by hand.

6. **Matrix Sketching**

Matrix sketching is an important technique in randomized linear algebra to do large computations efficiently. For example, to compute the multiplication $\mathbf{A}^T \times \mathbf{B}$ of two large matrices $\mathbf{A}$ and $\mathbf{B}$, we can use a random sketch matrix $\mathbf{S}$ to compute a "sketch" $\mathbf{SA}$ of $\mathbf{A}$ and a "sketch" $\mathbf{SB}$ of $\mathbf{B}$. Such a sketching matrix has the property that $\mathbf{S}^T\mathbf{S} \approx \mathbf{I}$ so that the approximate multiplication $\mathbf{A}^T\mathbf{S}^T\mathbf{SB}$ is close to $\mathbf{A}^T\mathbf{B}$.

In this problem, we will discuss two popular sketching schemes and understand how they help in approximate computation. Let $\hat{\mathbf{I}} = \mathbf{S}^T\mathbf{S}$ and the dimension of sketch matrix $\mathbf{S}$ be $d \times n$ (typically $d \ll n$).

(a) (**Gaussian-sketch**) Define

$$\mathbf{S} = \frac{1}{\sqrt{d}} \begin{bmatrix} S_{11} & \dots & \dots & S_{1n} \\ \vdots & \ddots & & \vdots \\ S_{d1} & \dots & \dots & S_{dn} \end{bmatrix}$$

such that S_{ij}'s are chosen i.i.d. from $\mathcal{N}(0,1)$ for all $i \in [1,d]$ and $j \in [1,n]$. Find the element-wise mean and variance (as a function of d) of the matrix $\hat{\mathbf{I}} = \mathbf{S}^T\mathbf{S}$, that is, find $\mathbb{E}[\hat{I}_{ij}]$ and $\mathrm{Var}[\hat{I}_{ij}]$ for all $i \in [1,n]$ and $j \in [1,n]$.

(b) (**Count-sketch**) For each column $j \in [1,n]$ of $\mathbf{S}$, choose a row i uniformly randomly from $[1,d]$ such that

$$S_{ij} = \begin{cases} 1, & \text{with probability } 0.5 \\ -1, & \text{with probability } 0.5 \end{cases}$$

and assign $S_{kj} = 0$ for all $k \neq i$. An example of a 3×8 count-sketch is

$$\mathbf{S} = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Again, find the element-wise mean and variance (as a function of d) of the matrix $\hat{\mathbf{I}} = \mathbf{S}^T\mathbf{S}$.

Note that for sufficiently large d, the matrix $\hat{\mathbf{I}}$ is close to the identity matrix for both cases. We will use this fact in the lab to do an approximate matrix multiplication.

Note: You can use the fact that the fourth moment of a standard Gaussian is 3 without proof.