UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 8
Fall 2020

1. **Poisson Branching**

   Consider a branching process such that at generation n, each individual in the population survives until generation $n+1$ with probability $0 < p < 1$, independently, and a Poisson number (with parameter λ) of immigrants enters the population. Let X_n denote the number of people in the population at generation n.

   (a) Suppose that at generation 0, the number of people in the population is a Poisson random variable with parameter λ_0. What is the distribution at generation 1? What is the distribution at generation n?

   (b) What is the distribution of X_n as $n \to \infty$? What if at generation 0, the number of individuals is an arbitrary probability distribution over the non-negative integers; does the distribution still converge? (Justify the convergence rigorously.)

   Hint: set up the problem as a Markov chain and argue convergence via the Big Theorem.

2. **Arrival Times of a Poisson Process**

   Consider a Poisson process $(N_t, t \geq 0)$ with rate $\lambda = 1$. For $i \in \mathbb{Z}_{>0}$, let S_i be a random variable which is equal to the time of the i-th arrival.

   (a) Find $\mathbb{E}[S_3 \mid N_1 = 2]$.

   (b) Given $S_3 = s$, where $s > 0$, find the joint distribution of S_1 and S_2.

   (c) Find $\mathbb{E}[S_2 \mid S_3 = s]$.

3. **Random Telegraph Wave**

   Let $\{N_t,\ t \geq 0\}$ be a Poisson process with rate λ and define $X_t = X_0(-1)^{N_t}$ where $X_0 \in \{0, 1\}$ is a random variable independent of N_t.

   (a) Does the process X_t have independent increments?

   (b) Calculate $\mathbb{P}(X_t = 1)$ if $\mathbb{P}(X_0 = 1) = p$.

   (c) Assume that $p = 0.5$. Calculate $\mathbb{E}[X_{t+s}X_s]$ for $s, t \geq 0$.

4. **Basketball II**

   Team A and Team B are playing an untimed basketball game in which the two teams score points according to independent Poisson processes. Team A scores points according to a Poisson process with rate λ_A and Team B scores points according to a Poisson process with rate λ_B. The game is over when one of the teams has scored k more points than the other team. Find the probability that Team A wins.

5. Illegal U-Turns

Each morning, as you pull out of your driveway, you would like to make a U-turn rather than drive around the block. Unfortunately, U-turns are illegal and police cars drive by according to a Poisson process with rate λ. You decide to make a U-turn once you see that the road has been clear of police cars for $\tau > 0$ units of time. Let N be the number of police cars you see before you make a U-turn.

(a) Find $\mathbb{E}[N]$.

(b) Let n be a positive integer ≥ 2. Find the conditional expectation of the time elapsed between police cars $n-1$ and n, given that $N \geq n$.

(c) Find the expected time that you wait until you make a U-turn.

6. Jukes-Cantor Model

In this question we consider a CTMC model for the evolution of DNA over time. Consider a CTMC $(X_t)_{t\geq 0}$ on the states $\mathcal{X} := \{A, C, G, T\}$ with transition rate matrix

$$Q = \begin{bmatrix} -3\lambda & \lambda & \lambda & \lambda \\ \lambda & -3\lambda & \lambda & \lambda \\ \lambda & \lambda & -3\lambda & \lambda \\ \lambda & \lambda & \lambda & -3\lambda \end{bmatrix}, \qquad \text{for } \lambda > 0.$$

Find the stationary distribution π and show it solves $\pi Q = 0$.

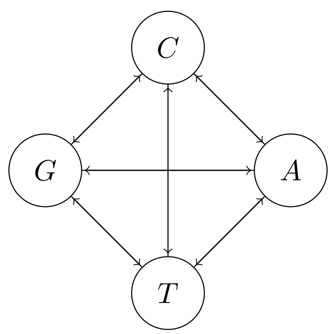

Figure 1: All edges in the rate diagram have rate λ.