UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 9
Fall 2020

1. Revisiting Basketball II

Last week we looked at modeling the probability of victory between two teams with scores according to a Poisson process. Depending on how detailed your justifications were last week, you may have already done this, but this is an exercise to double check you understood the formulation. Describe the following for this problem:

(a) Explicitly write out/draw the CTMC corresponding to this process.

(b) Explicitly write out/draw the DTMC that we used to model this process. Is this the embedded/jump chain or is it the uniformization chain? Does it matter which one we chose?

(c) In one or two sentences, describe why the DTMC we chose was a natural choice to model this process, even though it is originally a CTMC.

We copy the problem here for convenience: Team A and Team B are playing an untimed basketball game in which the two teams score points according to independent Poisson processes. Team A scores points according to a Poisson process with rate λ_A and Team B scores points according to a Poisson process with rate λ_B. The game is over when one of the teams has scored k more points than the other team. Find the probability that Team A wins.

2. Frogs

Three frogs are playing near a pond. When they are in the sun they get too hot and jump in the lake at rate 1. When they are in the lake they get too cold and jump onto the land at rate 2. The rates here refer to the rate in exponential distribution. Let X_t be the number of frogs in the sun at time $t \geq 0$.

(a) Find the stationary distribution for $(X_t)_{t\geq 0}$.

(b) Check the answer to (a) by noting that the three frogs are independent two-state Markov chains.

3. $M/M/2$ Queue

A queue has Poisson arrivals with rate λ. It has two servers that work in parallel. When there are at least two customers in the queue, two are being served. When there is only one customer, only one server is active. The service times are i.i.d. exponential random variables with rate μ. Let $X(t)$ be the number of customers either in the queue or in service at time t.

(a) Argue that the process $(X(t), t \geq 0)$ is a Markov process.

(b) Draw the state transition diagram.

(c) Find the range of values of μ for which the Markov chain is positive-recurrent and for this range of values calculate the stationary distribution of the Markov chain.

4. **CTMC Uniformization**

Consider the CTMC defined with state space $\{1, 2, 3\}$ and the rate matrix

$$Q = \begin{bmatrix} -4 & 1 & 3 \\ 0 & -3 & 3 \\ 1 & 1 & -2 \end{bmatrix}.$$

(a) Let $q = 5$. Compute the transition matrix for the uniformized chain as $P = I + \frac{1}{q}Q$.

(b) Describe what q represents.

(c) Why must $q \geq \max_i q_i$?

(d) Inuitively, why does the probability of a self-jump increase when q increases?

5. **Bus Arrivals at Cory Hall**

Starting at time 0, the 52 line makes stops at Cory Hall according to a Poisson process of rate λ. Students arrive at the stop according to an independent Poisson process of rate μ. Every time the bus arrives, all students waiting get on.

(a) Given that the interarrival time between bus $i-1$ and bus i is x, find the distribution for the number of students entering the ith bus. (Here, x is a given number, not a random quantity.)

(b) Given that a bus arrived at 9:30 AM, find the distribution for the number of students that will get on the next bus.

(c) Find the distribution of the number of students getting on the next bus to arrive after 9:30 AM, assuming that time 0 was infinitely far in the past.

6. **Entropy of a Sum**

Let X_1, X_2 be i.i.d. Bernoulli(1/2) (fair coin flips). Calculate $H(X_1 + X_2)$ and show that $H(X_1 + X_2) \geq H(X_1)$. In fact it is generally true that adding independent random variables increases entropy.

Note: It is known that the Gaussian distribution maximizes entropy given a constraint on the variance. Therefore, one intuitive interpretation of the CLT is that convolving independent random variables tends to increase uncertainty until the sum approaches the distribution which "maximizes uncertainty", the Gaussian distribution. Proving the CLT along these lines is far from easy, however.