UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

**Discussion 13**
Fall 2021

1. **Orthogonal LLSE**

   (a) Consider zero-mean random variables $X, Y, Z$ such that $Y, Z$ are orthogonal. Show that $L[X \mid Y, Z] = L[X \mid Y] + L[X \mid Z]$.

   (b) Explain why for any zero-mean random variables $X, Y, Z$ it holds that:

$$L[X \mid Y, Z] = L[X \mid Y] + L\big[X \bigm| Z - L[Z \mid Y]\big]$$

2. **MMSE for Jointly Gaussian Random Variables**

   Provide justification for each of the following steps (1 - 5) to prove that the LLSE is equal to the MMSE estimator for jointly Gaussian random variables $X$ and $Y$.
   Let $g(X) = L[Y \mid X]$.

$$\begin{aligned} & E[(Y - g(X))X] = 0 && (1) \\ \implies & \operatorname{cov}(Y - g(X), X) = 0 && (2) \\ \implies & Y - g(X) \text{ is independent of } X && (3) \\ \implies & E[(Y - g(X))f(X)] = 0 \ \forall f(\cdot) && (4) \\ \implies & g(X) = E[Y \mid X] && (5) \end{aligned}$$

3. **Stochastic Linear System MMSE**

   Let $(V_n,\ n \in \mathbb{N})$ be i.i.d. $\mathcal{N}(0, \sigma^2)$ and independent of $X_0 = \mathcal{N}(0, u^2)$. Let $|a| < 1$. Define

$$X_{n+1} = aX_n + V_n, \qquad n \in \mathbb{N}.$$

   (a) What is the distribution of $X_n$, where $n$ is a positive integer?

   (b) Find $\mathbb{E}[X_{n+m} \mid X_n]$ for $m, n \in \mathbb{N}$, $m \geq 1$.

   (c) Find $u$ so that the distribution of $X_n$ is the same for all $n \in \mathbb{N}$.

4. **(Optional, included for practice) Random Walk with Unknown Drift**

   Consider a random walk with unknown drift. The dynamics are given, for $n \in \mathbb{N}$, as

$$\begin{aligned} X_1(n+1) &= X_1(n) + X_2(n) + V(n), \\ X_2(n+1) &= X_2(n), \\ Y(n) &= X_1(n) + W(n). \end{aligned}$$

   Here, $X_1$ represents the position of the particle and $X_2$ represents the velocity of the particle (which is unknown but constant throughout time). $Y$ is the observation. $V$ and $W$ are independent Gaussian noise variables with mean zero and variance $\sigma_V^2$ and $\sigma_W^2$ respectively.

(a) Write down the dynamics of the system in matrix-vector form and write down the Kalman filter recursive equations for this system.

(b) Let $k$ be a positive integer. Compute the prediction $\mathbb{E}(X(n+k) \mid Y^{(n)})$, where $Y^{(n)}$ is the history of the observations $Y_0, \ldots, Y_n$, in terms of the estimate $\hat{X}(n) := \mathbb{E}(X(n) \mid Y^{(n)})$.

(c) Now let $k = 1$ and compute the smoothing estimate $\mathbb{E}(X(n) \mid Y^{(n+1)})$ in terms of the quantities that appear in the Kalman filter equation.

*Hint:* Use the law of total expectation

$$\mathbb{E}\big(X(n) \mid Y^{(n+1)}\big) = \mathbb{E}\big[\mathbb{E}\big(X(n) \mid X(n+1), Y^{(n+1)}\big) \mid Y^{(n+1)}\big],$$

as well as the *innovation*

$$\tilde{X}(n+1) := X(n+1) - L[X(n+1) \mid Y^{(n)}].$$