UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 5
Fall 2021

1. Curse of Dimensionality

In this problem, we will use the law of large numbers to illustrate a statistical phenomenon. In particular, consider the hypercube $[-1,1]^n$ in $\mathbb{R}^n$, and let $X_1, \ldots, X_n$ be iid Uniform($[-1,1]$).

(a) For $\epsilon > 0$ consider the set

$$A_{n,\epsilon} := \{x \in \mathbb{R}^n : (1-\epsilon)\sqrt{n/3} < \|x\|_2 < (1+\epsilon)\sqrt{n/3}\},$$

which is the ϵ-boundary of a ball with radius $\sqrt{n/3}$ centered at the origin. For low dimensions $n = 1, 2$ and $\epsilon = 1/10$, compute the fraction of volume of $[-1,1]^n$ which comes from $A_{n,\epsilon}$.

(b) Show that as n gets large, most of the volume of the hypercube comes from $A_{n,\epsilon}$. Comment on why this contradicts the intuition developed in part (a).

2. Product of Rolls of a Die

A fair die with labels (1 to 6) is rolled until the product of the last two rolls is 12. What is the expected number of rolls?

[Hint: You can model this process as a Markov chain with 3 states. Choose your states according to the outcome of last roll. For example, assign one state if it is outcome was 1 or 5 (which is useless if you want the product to be 12). If the outcome was 2,3,4 or 6, it's useful and can be assigned another state. Assign third state to the case when the product last two outcomes was 12.]

3. Concentration for Binomials & Gaussians

For sums of bounded zero-mean i.i.d. random variables $S_n = X_1 + \ldots + X_n$, a Chernoff-type inequality tells us that

$$P(|S_n| \geq t\sqrt{n}) \leq C\exp(-ct^2),$$

for some constants $C, c > 0$.

(a) (Optional) Prove the inequality above.

(b) Let $Z \sim \mathcal{N}(0,1)$. Show that

$$P(|Z| \geq t) \leq C\exp(-ct^2).$$